RANDOM TRIGONOMETRIC POLYNOMIALS

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Borwein and Lockhart [1].

1. L^2 norm

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and for $f : \mathbb{T} \to \mathbb{C}$ let

$$\|f\|_{L^p}^p = \int_0^1 |f(\theta)|^p d\theta.$$

Let $X_1, X_2, \dots : (\Omega, \mathscr{F}, P) \to (\mathbb{R}, \mathscr{B}_\mathbb{R})$ be independent identically distributed random variables with mean 0 and variance 1, and define

$$q_n(\theta) = \sum_{j=0}^{n-1} X_j e^{2\pi i j \theta}, \qquad \theta \in \mathbb{T}.$$

By Plancherel's theorem,

$$\|q_n\|_{L^2}^2 = \sum_{j=0}^{n-1} X_j^2.$$

Let $Y_j = X_j^2 - 1$, which are independent and identically distributed. Then

$$\|q_n\|_{L^2}^2 - n = \sum_{j=0}^{n-1} Y_j.$$

We have

$$E(Y_j) = E(X_j^2) - 1 = 0.$$

Write

$$\sigma^2 = E(Y_j^2) = E(X_j^4 - 2X_j^2 + 1) = E(X_j^4) - 2E(X_j^2) + 1 = E(X_j^4) - 1,$$

and let

$$Z_n = \frac{\sum_{j=0}^{n-1} Y_j}{\sigma\sqrt{n}} = \frac{\|q_n\|_{L^2}^2 - n}{\sigma\sqrt{n}},$$

which has mean 0 and variance 1. Because $Y_1, Y_2, \dots$ are independent and identically distributed with mean 0 and variance σ^2, by the central limit theorem, $Z_n \to \gamma_1$ in distribution, where γ_{t^2} is the Gaussian measure on $\mathbb{R}$ with variance t^2.

Theorem 1.

$$E(\|q_n\|_{L^2}) = \sqrt{n} - \frac{1}{8}\frac{\sigma^2}{\sqrt{n}} + O(n^{-1}),$$

where $\sigma^2 = E(X_j^4) - 1$.

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Proof. Because

$$\|q_n\|_{L^2} = \sqrt{n + \sigma\sqrt{n} Z_n},$$

we have

$$\|q_n\|_{L^2} - \sqrt{n} = \sqrt{n}\left(\sqrt{1 + \frac{\sigma Z_n}{\sqrt{n}}} - 1\right).$$

Using the binomial series,

$$\sqrt{1 + \frac{\sigma Z_n}{\sqrt{n}}} = 1 + \frac{\sigma Z_n}{2\sqrt{n}} - \frac{1}{8}\frac{\sigma^2 Z_n^2}{n} + O\left(\frac{Z_n^3}{n^{3/2}}\right),$$

so

$$\|q_n\|_{L^2} - \sqrt{n} = \frac{\sigma Z_n}{2} - \frac{1}{8}\frac{\sigma^2 Z_n^2}{\sqrt{n}} + O\left(\frac{Z_n^3}{n}\right). \tag{1}$$

We expand Z_n^4: it is

$$\sigma^{-4} n^{-2}\left(\sum_j Y_j^4 + \sum_{j\neq k} Y_j^3 Y_k + \sum_{j \neq k} Y_j^2 Y_k^2 + \sum_{j\neq k\neq p} Y_k^2 Y_j Y_p + \sum_{j\neq k\neq p\neq q} Y_j Y_k Y_p Y_q\right).$$

Then, because $E(Y_j) = 0$ and $E(Y_j^2) = \sigma^2$, we get

$$E(Z_n^4) = \sigma^{-4} n^{-2}(nE(Y_1^4) + n(n-1)\sigma^4).$$

Now define

$$\tau = E(Y_j^4),$$

so

$$E(Z_n^4) = \sigma^{-4} n^{-2}(n\tau + n(n-1)\sigma^4) = 1 + \frac{1}{n}\left(\frac{\tau}{\sigma^4} - 1\right).$$

But $E(|Z_n|^3)^{1/3} \leq E(|Z_n|^4)^{1/4}$, so

$$E(|Z_n|^3) \leq \left(1 + \frac{1}{n}\left(\frac{\tau}{\sigma^4} - 1\right)\right)^{3/4} \leq 1.$$

Taking the expectation of (1), because $E(Z_n) = 0$ and $E(Z_n^2) = 1$,

$$E(\|q_n\|_{L^2}) = \sqrt{n} - \frac{1}{8}\frac{\sigma^2}{\sqrt{n}} + O(n^{-1}).$$

□

2. Berry-Esseen

Theorem 2.

$$\|q_n\|_{L^2} - \sqrt{n} \to \frac{\sigma}{2} Z$$

in distribution.

Proof. Write

$$\rho = E(|Y_j|^3)$$

and

$$S_n = \sum_{j=0}^{n-1} Y_j,$$

and let

$$F_n(x) = P(S_n \leq \sigma n^{1/2} x)$$

and

$$\Phi(x) = P(Z \leq x).$$

The Berry-Esseen theorem [2, p. 262, Theorem 5.6.1] states that there is some C, not depending on the random variables Y_j, such that for all n and for all $x \in \mathbb{R}$,

$$|F_n(x) - \Phi(x)| \leq \frac{C\rho}{\sigma^3 \sqrt{n}}.$$

Now,

$$Z_n = \frac{1}{\sigma n^{1/2}} \sum_{j=0}^{n-1} Y_j = \frac{S_n}{\sigma n^{1/2}},$$

so

$$F_n(x) = P\left(\frac{S_n}{\sigma n^{1/2}} \leq x\right) = P(Z_n \leq x).$$

For $A > 0$,

$$P(|Z_n| \geq A) = P(Z_n \geq A) + P(Z_n \leq -A) = 1 - P(Z_n < A) + P(Z_n \leq -A).$$

$$P(|Z_n| \geq A) - P(|Z| \geq A) = P(Z < A) - P(Z_n < A) + P(Z_n \leq_A) - P(Z \leq -A).$$

Then

$$|P(|Z_n| \geq A) - P(|Z| \geq A)| \leq |\Phi(A) - F_n(A)| + P(Z_n = A) + |F_n(-A) - \Phi(-A)|,$$

so by the Berry-Esseen theorem,

$$|P(|Z_n| \geq A) - P(|Z| \geq A)| \leq P(Z_n = A) + 2\frac{C\rho}{\sigma^3 n^{1/2}}.$$

Markov's inequality tells us

$$P(|Z| \geq A) \leq \sqrt{\frac{2}{\pi}} \frac{e^{-A^2/2}}{A}.$$

For $\epsilon > 0$ and for $A = n^{1/4} e^{1/2}$,

$$P(|Z| \geq A) = P(|Z|^2 \geq n^{1/2}\epsilon) \leq \frac{\sqrt{\frac{2}{\pi}} \exp\left(-\frac{n^{1/2}\epsilon}{2}\right)}{n^{1/4}\epsilon^{1/2}}.$$

Therefore $\frac{Z_n^2}{n^{1/2}} \to 0$ in probability and $\frac{|Z_n|^3}{n} \to 0$ in probability, and because $\frac{\sigma Z_n}{2} \to \frac{\sigma}{2} Z$ in distribution, it follows that

$$\|q_n\|_{L^2} - \sqrt{n} \to \frac{\sigma}{2} Z$$

in distribution. □

3. L^4 NORM

Theorem 3.
$$E(\|q_n\|_{L^4}^4) = 2n^2 + n(E(X_j^4) - 2).$$

Proof.
$$\|q_n\|_{L^4}^4 = \int_0^1 q_n(\theta)^2 \overline{q_n(\theta)^2} d\theta.$$

Write $e(\theta) = e^{2\pi i\theta}$.

$$q_n^2 = \sum_j X_j^2 e(2j\theta) + \sum_{j\neq k} X_j X_k e(j\theta + k\theta).$$

$$\begin{aligned} q_n^2\overline{q_n^2} &= \sum_{j,p} X_j^2 e(2k\theta) X_p^2 e(-2p\theta) + \sum_p X_p^2 e(-2p\theta) \sum_{j\neq k} X_j X_k e(k\theta + j\theta) \\ &+ \sum_{p\neq q} X_p X_q e(-p\theta - q\theta) \sum_j X_j^2 e(2j\theta) \\ &+ \sum_{p\neq q} X_p X_q e(-p\theta - q\theta) \sum_{j\neq k} X_j X_k e(k\theta + j\theta). \end{aligned}$$

Then

$$\begin{aligned} \int_0^1 q_n^2\overline{q_n^2} d\theta &= \sum_j \sum_p X_j^2 X_p^2 \delta_{j-p,0} \\ &+ \sum_p \sum_{j\neq k} X_p^2 X_j X_k \delta_{j+k-2p,0} \\ &+ \sum_j \sum_{p\neq q} X_j^2 X_p X_q \delta_{2j-p-q,0} \\ &+ \sum_{j\neq k} \sum_{p\neq q} X_j X_k X_p X_q \delta_{j+k-p-q,0}. \end{aligned}$$

That is

$$\begin{aligned} \|q_n\|_{L^4}^4 &= \sum_j \sum_p X_j^2 X_p^2 \delta_{j,p} \\ &+ \sum_p \sum_{j\neq k} X_p^2 X_j X_k \delta_{j+k,2p} \\ &+ \sum_j \sum_{p\neq q} X_j^2 X_p X_q \delta_{p+q,2j} \\ &+ \sum_{j\neq k} \sum_{p\neq q} X_j X_k X_p X_q \delta_{j+k,p+q} \\ &= \sum_j X_j^4 + 2\sum_p \sum_{j\neq p} \sum_{k\neq p, k\neq j} X_p^2 X_j X_k \delta_{j+k,2p} \\ &+ \sum_j \sum_{k\neq j} \sum_{p\neq j, p\neq k} \sum_{q\neq p, q\neq j, q\neq k} X_j X_k X_p X_q \delta_{j+k,p+q} \\ &+ \sum_j \sum_{k\neq j} X_j^2 X_k^2 + \sum_j \sum_{k\neq j} X_j^2 X_k^2. \end{aligned}$$

Then

$$E(\|q_n\|_{L^4}^4) = \sum_j E(X_j^4) + 2\sum_j \sum_{k\neq j} E(X_j^2)E(X_k^2)$$
$$= nE(X_j^4) + 2n^2 - 2n.$$

□

4. Gaussian random variables

Suppose that the distribution of each X_j is the standard Gaussian measure on $\mathbb{R}$, and write

$$S_{n,\theta} = \sum_{j=0}^{n-1} X_j e^{2\pi i j\theta} = \sum_{j=0}^{n-1} X_j \cos 2\pi j\theta + i\sum_{j=0}^{n-1} X_j \sin 2\pi j\theta, \quad n \geq 1, \quad \theta \in \mathbb{T}.$$

Then for each $\theta \in \mathbb{T}$, there are Z_θ and W_θ, each random variables with the standard Gaussian distribution, such that

$$S_{n,\theta} = (n/2)^{1/2} Z_\theta + i(n/2)^{1/2} W_\theta.$$

Now, $|Z_\theta + iW_\theta|$ has density $t \mapsto te^{-t^2/2}$, and then

$$E(|Z_\theta + iW_\theta|^p) = 2^{p/2}\Gamma\left(1 + \frac{p}{2}\right).$$

Then

$$E(|S_{n,\theta}|^p) = \left(\frac{n}{2}\right)^{p/2} 2^{p/2}\Gamma\left(1 + \frac{p}{2}\right) = n^{p/2}\Gamma\left(1 + \frac{p}{2}\right).$$

References

1. Peter Borwein and Richard Lockhart, *The expected L_p norm of random polynomials*, Proc. Amer. Math. Soc. **129** (2001), no. 5, 1463–1472.
2. Mark A. Pinsky, *Introduction to Fourier analysis and wavelets*, Graduate Studies in Mathematics, vol. 102, American Mathematical Society, Providence, RI, 2009.

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