

# Chebyshev polynomials

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## 1 $T_n$

On the one hand,

$$\begin{aligned}(\cos \theta + i \sin \theta)^n &= \sum_{0 \leq \nu \leq n} i^\nu \binom{n}{\nu} \cos^{n-\nu} \theta \sin^\nu \theta \\ &= \sum_{0 \leq 2k \leq n} (-1)^k \binom{n}{2k} \cos^{n-2k}(\theta) \sin^{2k}(\theta) \\ &\quad + i \sum_{0 \leq 2k+1 \leq n} (-1)^k \binom{n}{2k+1} \cos^{n-2k-1}(\theta) \sin^{2k+1}(\theta).\end{aligned}$$

On the other hand,

$$(\cos \theta + i \sin \theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i \sin n\theta.$$

Therefore

$$\begin{aligned}\cos n\theta &= \sum_{0 \leq k \leq n/2} (-1)^k \binom{n}{2k} \cos^{n-2k}(\theta) \sin^{2k}(\theta) \\ &= \sum_{0 \leq k \leq n/2} (-1)^k \binom{n}{2k} \cos^{n-2k}(\theta) (1 - \cos^2 \theta)^k \\ &= \sum_{0 \leq k \leq n/2} \binom{n}{2k} \cos^{n-2k}(\theta) (\cos^2 \theta - 1)^k \\ &= \sum_{0 \leq k \leq n/2} \binom{n}{2k} \cos^{n-2k}(\theta) \sum_{0 \leq j \leq k} \binom{k}{j} \cos^{2k-2j}(\theta) (-1)^j \\ &= \sum_{0 \leq j \leq n/2} (-1)^j \cos^{n-2j}(\theta) \sum_{j \leq k \leq n/2} \binom{n}{2k} \binom{k}{j}.\end{aligned}$$

Now,

$$\sum_{j \leq k \leq n/2} \binom{n}{2k} \binom{k}{j} = 2^{n-2j-1} \binom{n-j}{j} \frac{n}{n-j}.$$

Hence

$$\cos n\theta = \sum_{0 \leq j \leq n/2} (-1)^j \cos^{n-2j}(\theta) 2^{n-2j-1} \binom{n-j}{j} \frac{n}{n-j}.$$

For  $z \in \mathbb{C}$  let

$$T_n(z) = \sum_{0 \leq j \leq n/2} (-1)^j 2^{n-2j-1} \binom{n-j}{j} \frac{n}{n-j} z^{n-2j}. \quad (1)$$

Note

$$T_n(z)[z^n] = 2^{n-1} z^n.$$

**Theorem 1.**

$$T_n(\cos \theta) = \cos(n\theta)$$

and

$$T_m \circ T_n = T_{mn}.$$

*Proof.* For  $\theta \in \mathbb{R}$ ,

$$T_n(\cos \theta) = \cos(n\theta).$$

Then

$$T_m(T_n(\cos \theta)) = T_m(\cos(n\theta)) = \cos(mn\theta) = T_{mn}(\theta).$$

That is, for  $z \in [-1, 1]$  we have  $T_m(T_n(z)) = T_{mn}(z)$ . Then by analytic continuation it follows that this is true for all  $z$ .  $\square$

**Theorem 2.**

$$T_n(z) + T_{n-2}(z) = 2zT_{n-1}(z).$$

*Proof.* Using  $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$ ,

$$\cos(n\theta) = \cos(\theta + (n-1)\theta) = \cos \theta \cos((n-1)\theta) - \sin \theta \sin((n-1)\theta)$$

and

$$\cos((n-2)\theta) = \cos(-\theta + (n-1)\theta) = \cos \theta \cos((n-1)\theta) + \sin \theta \sin((n-1)\theta).$$

Then

$$\cos(n\theta) + \cos((n-2)\theta) = 2 \cos \theta \cos((n-1)\theta).$$

Therefore

$$\begin{aligned} T_n(\cos \theta) + T_{n-2}(\cos \theta) &= \cos(n\theta) + \cos((n-2)\theta) \\ &= 2 \cos \theta \cos((n-1)\theta) \\ &= 2 \cos \theta \cdot T_{n-1}(\cos \theta). \end{aligned}$$

That is, for  $z \in [-1, 1]$ ,

$$T_n(z) + T_{n-2}(z) = 2zT_{n-1}(z),$$

and by analytic continuation this is true for all  $z \in \mathbb{C}$ .  $\square$

## 2 $U_n$

Define

$$nU_{n-1}(z) = T'_n(z). \quad (2)$$

**Theorem 3.**

$$U_{n-1}(\cos \theta) = \frac{\sin(n\theta)}{\sin \theta}$$

and

$$(1 - z^2)T''_n(z) = nU_n(z) - n(n+1)T_n(z).$$

*Proof.* On the one hand,

$$(T_n(\cos \theta))' = -\sin \theta \cdot T'_n(\cos \theta).$$

On the other hand,

$$(T_n(\cos \theta))' = (\cos(n\theta))' = -n \sin(n\theta).$$

Hence

$$T'_n(\cos \theta) = n \frac{\sin(n\theta)}{\sin \theta}, \quad U_{n-1}(\cos \theta) = \frac{\sin(n\theta)}{\sin \theta}.$$

Now,

$$(T'_n(\cos \theta))' = -\sin \theta \cdot T''_n(\cos \theta)$$

and

$$\begin{aligned} (T'_n(\cos \theta))' &= n \frac{n \cos(n\theta) \sin \theta - \sin(n\theta) \cos \theta}{\sin^2 \theta} \\ &= -n \frac{\cos(n\theta) \sin \theta + \sin(n\theta) \cos \theta}{\sin^2 \theta} + n(n+1) \frac{\cos(n\theta) \sin \theta}{\sin^2 \theta} \\ &= -n \frac{\sin((n+1)\theta)}{\sin^2 \theta} + n(n+1) \frac{\cos(n\theta)}{\sin \theta} \\ &= -n \frac{U_n(\cos \theta)}{\sin \theta} + n(n+1) \frac{T_n(\cos \theta)}{\sin \theta}. \end{aligned}$$

Hence

$$T''_n(\cos \theta) = n \frac{U_n(\cos \theta)}{\sin^2 \theta} - n(n+1) \frac{T_n(\cos \theta)}{\sin^2 \theta}$$

and then

$$T''_n(\cos \theta) = n \frac{U_n(\cos \theta)}{1 - \cos^2 \theta} - n(n+1) \frac{T_n(\cos \theta)}{1 - \cos^2 \theta}.$$

By analytic continuation,

$$(1 - z^2)T_n''(z) = nU_n(z) - n(n + 1)T_n(z).$$

□

**Theorem 4.**

$$T_{n+1}(z) = zT_n(z) - (1 - z^2)U_{n-1}(z).$$

*Proof.*

$$\begin{aligned} T_{n+1}(\cos \theta) &= \cos(n\theta + \theta) \\ &= \cos(n\theta) \cos \theta - \sin(n\theta) \sin \theta \\ &= T_n(\cos \theta) \cos \theta - U_{n-1}(\cos \theta) \sin^2 \theta \\ &= T_n(\cos \theta) \cos \theta - U_{n-1}(\cos \theta)(1 - \cos^2 \theta). \end{aligned}$$

Therefore by analytic continuation,

$$T_{n+1}(z) = zT_n(z) - (1 - z^2)U_{n-1}(z).$$

□

**Theorem 5.**

$$U_n(z) = T_n(z) + zU_{n-1}(z).$$

*Proof.*

$$\begin{aligned} U_n(\cos \theta) &= \frac{\sin(n\theta + \theta)}{\sin \theta} \\ &= \frac{\cos(n\theta) \sin \theta + \cos \theta \sin(n\theta)}{\sin \theta} \\ &= T_n(\cos \theta) + \cos \theta \cdot U_{n-1}(\cos \theta). \end{aligned}$$

Therefore by analytic continuation,

$$U_n(z) = T_n(z) + zU_{n-1}(z).$$

□

**Theorem 6.**

$$U_n(z) = 2zU_{n-1}(z) + U_{n-2}(z).$$

*Proof.* Using Theorem 4 and Theorem 5,

$$\begin{aligned}
U_n(z) &= T_n(z) + zU_{n-1}(z) \\
&= zT_{n-1}(z) - (1 - z^2)U_{n-2}(z) + zU_{n-1}(z) \\
&= z \left[ U_{n-1}(z) - zU_{n-2}(z) \right] - (1 - z^2)U_{n-2}(z) + zU_{n-1}(z) \\
&= 2zU_{n-1}(z) + U_{n-2}(z).
\end{aligned}$$

□

**Theorem 7.**

$$(1 - z^2)T_n''(z) - zT_n'(z) + n^2T_n(z) = 0.$$

*Proof.* Using Theorem 3, and Theorem 5,

$$\begin{aligned}
&(1 - z^2)T_n''(z) - zT_n'(z) + n^2T_n(z) \\
&= nU_n(z) - n(n+1)T_n(z) - nzU_{n-1}(z) + n^2T_n(z) \\
&= n(T_n(z) + zU_{n-1}(z)) - n(n+1)T_n(z) - nzU_{n-1}(z) + n^2T_n(z) \\
&= 0.
\end{aligned}$$

□

From Theorem 1

$$T_n(1) = T_n(\cos 0) = \cos(n \cdot 0) = 1.$$

From Theorem 3,

$$T_n'(1) = nU_{n-1}(1) = n^2.$$

Thus,  $T_n$  is the unique solution of the initial value problem

$$(1 - x^2)y''(x) - xy'(x) + n^2y(x) = 0, \quad y(1) = 1, y'(1) = n^2.$$

**Theorem 8.**

$$T_n(z)^2 - (z^2 - 1)U_{n-1}(z)^2 = 1.$$

*Proof.* Using Theorem 1 and Theorem 3, for  $z = \cos \theta$ ,

$$\begin{aligned}
T_n(z)^2 - (z^2 - 1)U_{n-1}(z)^2 &= T_n(\cos \theta)^2 + (\sin^2 \theta)U_{n-1}(\cos \theta)^2 \\
&= \cos^2(n\theta) + (\sin^2 \theta) \frac{\sin^2(n\theta)}{\sin^2 \theta} \\
&= \cos^2(n\theta) + \sin^2(n\theta) \\
&= 1.
\end{aligned}$$

By analytic continuation, this is true for all  $z$ .

□

### 3 Inner products

For  $0 \leq \theta \leq \pi$  let  $y_n(\theta) = \cos(n\theta)$ .

$$y_n'' + n^2 y_n = 0, \quad y_n'(0) = 0, y_n'(\pi) = 0.$$

**Theorem 9.**

$$\int_{-1}^1 \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}} dx = \int_0^\pi y_m y_n d\theta = \frac{\pi}{2} \cdot \delta_{m,n}.$$

*Proof.* Let  $W = y_m y_n' - y_n y_m'$ . We calculate

$$\begin{aligned} W' &= y_m' y_n' + y_m y_n'' - y_n' y_m' - y_n y_m'' y_m y_n'' - y_n y_m'' \\ &= y_m y_n'' - y_n y_m'' \\ &= y_m(-n^2 y_n) - y_n(-m^2 y_m) \\ &= (m^2 - n^2) y_m y_n. \end{aligned}$$

Using  $W(0) = 0$  and  $W(\pi) = 0$ ,

$$\int_0^\pi W'(\theta) d\theta = W(\pi) - W(0) = 0.$$

Then

$$\int_0^\pi (m^2 - n^2) y_m y_n d\theta = 0.$$

Doing the substitution  $\phi = n\theta$ ,

$$\begin{aligned} \int_0^\pi y_n^2 d\theta &= \int_0^\pi \cos^2(n\theta) d\theta \\ &= \int_0^\pi \frac{1 + \cos(2n\theta)}{2} d\theta \\ &= \frac{\pi}{2}. \end{aligned}$$

Therefore

$$\int_0^\pi y_m y_n d\theta = \frac{\pi}{2} \cdot \delta_{m,n}.$$

For  $0 \leq \theta \leq \pi$ ,  $\sqrt{1 - \cos^2 \theta} = \sin \theta$ . Then doing the substitution  $x = \cos \theta$ ,

$$dx = -\sin \theta d\theta,$$

$$\begin{aligned}\int_0^\pi y_m y_n d\theta &= \int_0^\pi \cos(m\theta) \cos(n\theta) d\theta \\ &= \int_0^\pi \cos(m\theta) \cos(n\theta) \frac{-\sin \theta d\theta}{-\sin \theta} \\ &= \int_0^\pi \frac{\cos(m\theta) \cos(n\theta)}{-\sqrt{1-\cos^2 \theta}} (-\sin \theta) d\theta \\ &= \int_1^{-1} \frac{T_m(x) T_n(x)}{-\sqrt{1-x^2}} dx \\ &= \int_{-1}^1 \frac{T_m(x) T_n(x)}{\sqrt{1-x^2}} dx.\end{aligned}$$

□