

# ALTERNATING MULTILINEAR FORMS

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## 1. PERMUTATIONS

We follow Cartan [2] and Abraham and Marsden [1].

Let  $E$  be a real vector space. Let  $\mathcal{L}_p(E; \mathbb{R})$  be the set of multilinear maps  $E^p \rightarrow \mathbb{R}$ .

**Definition 1.** A map  $f \in \mathcal{L}_p(E; \mathbb{R})$  is called **alternating** if  $(x_1, \dots, x_p) \in E^p$  with  $x_i = x_{i+1}$  for some  $1 \leq i < p$  implies  $f(x_1, \dots, x_p) = 0$ . Let  $\mathcal{A}_p(E; \mathbb{R})$  be the set of alternating elements of  $\mathcal{L}_p(E; \mathbb{R})$ .

Let  $S_p$  be the group of permutations of  $\{1, \dots, p\}$ , i.e. bijections  $\{1, \dots, p\} \rightarrow \{1, \dots, p\}$ . For  $\sigma, \tau \in S_p$ , write  $\sigma\tau = \sigma \circ \tau$ .

**Definition 2.** For a function  $f : E^p \rightarrow \mathbb{R}$  and a permutation  $\sigma \in S_p$ , define the function  $\sigma f : E^p \rightarrow \mathbb{R}$  by

$$(\sigma f)(x_1, \dots, x_p) = f(x_{\sigma(1)}, \dots, x_{\sigma(p)}), \quad (x_1, \dots, x_p) \in E^p.$$

**Theorem 3.** For a function  $f : E^p \rightarrow \mathbb{R}$  and for  $\sigma, \tau \in S_p$ ,

$$\tau(\sigma f) = (\tau\sigma)f.$$

*Proof.* Define  $g = \sigma f$ . For  $(x_1, \dots, x_p) \in E^p$  and for  $y_i = x_{\tau(i)}$ , we have

$$\begin{aligned} \tau(\sigma f)(x_1, \dots, x_p) &= \tau(g)(x_1, \dots, x_p) \\ &= g(x_{\tau(1)}, \dots, x_{\tau(p)}) \\ &= g(y_1, \dots, y_p) \\ &= (\sigma f)(y_1, \dots, y_p) \\ &= f(y_{\sigma(1)}, \dots, y_{\sigma(p)}) \\ &= f(x_{\tau(\sigma(1))}, \dots, x_{\tau(\sigma(p))}) \\ &= (\tau\sigma)(f)(x_1, \dots, x_p). \end{aligned}$$

Thus

$$\tau(\sigma f) = (\tau\sigma)f.$$

□

For  $1 \leq i, j \leq p$ , define  $(i, j) \in S_p$  by

$$(i, j)(k) = \begin{cases} j & k = i, \\ i & k = j, \\ k & k \neq i, j, \end{cases}$$

called a **transposition**. Define

$$\tau_i = (i, i + 1),$$

called an **adjacent transposition**. We can write a transposition  $(i, j)$ ,  $i < j$ , as a product of  $2j - 2i - 1$  adjacent transpositions:

$$\begin{aligned} (i, j) &= (j - 1, j)(j - 2, j - 1) \cdots (i + 1, i + 2)(i, i + 1)(i + 1, i + 2) \cdots (j - 1, j) \\ &= \tau_{j-1} \cdots \tau_{i+1} \tau_i \tau_{i+1} \cdots \tau_{j-1}. \end{aligned}$$

**Theorem 4.** For  $\sigma, \tau \in S_p$ ,

$$\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau).$$

**Theorem 5.** Let  $f \in \mathcal{L}_p(E; \mathbb{R})$ .  $f \in \mathcal{A}_p(E; \mathbb{R})$  if and only if  $\sigma f = (\operatorname{sgn} \sigma)f$  for all  $\sigma \in S_p$ .

*Proof.* (i) Suppose that  $f \in \mathcal{A}_p(E; \mathbb{R})$  and let  $\sigma \in S_p$ ; we have to show that  $\sigma f = (\operatorname{sgn} \sigma)f$ . Let  $(x_1, \dots, x_p) \in E^p$  and for  $1 \leq i < p$  define  $g_i : E^2 \rightarrow \mathbb{R}$  by

$$g_i(y_1, y_2) = f(x_1, \dots, \underbrace{y_1}_i, \underbrace{y_2}_{i+1}, \dots, x_p), \quad (y_1, y_2) \in E^2.$$

Because  $f$  is multilinear and alternating, on the one hand

$$g_i(x_i + x_{i+1}, x_i + x_{i+1}) = 0,$$

and on the other hand

$$\begin{aligned} g_i(x_i + x_{i+1}, x_i + x_{i+1}) &= g_i(x_i, x_i) + g_i(x_i, x_{i+1}) + g_i(x_{i+1}, x_i) + g_i(x_{i+1}, x_{i+1}) \\ &= g_i(x_i, x_{i+1}) + g_i(x_{i+1}, x_i). \end{aligned}$$

Therefore

$$g_i(x_{i+1}, x_i) = -g_i(x_i, x_{i+1}),$$

that is,

$$f(x_1, \dots, x_p) = -f(x_1, \dots, x_p).$$

Thus, as  $\operatorname{sgn} \tau_i = -1$ ,

$$\tau_i f = (\operatorname{sgn} \tau_i) f.$$

Because  $\sigma$  is equal to a product of adjacent transpositions, it then follows from Theorem 3 and Theorem 4 that  $\sigma f = (\operatorname{sgn} \sigma)f$ .

(ii) Suppose that  $\sigma f = (\operatorname{sgn} \sigma)f$  for all  $\sigma \in S_p$ . Let  $(x_1, \dots, x_p) \in E^p$  with  $x_i = x_{i+1}$  for some  $1 \leq i < p$ ; we have to show that  $f(x_1, \dots, x_p) = 0$ . On the one hand,

$$\tau_i f(x_1, \dots, x_p) = (\operatorname{sgn} \tau_i) f(x_1, \dots, x_p) = -f(x_1, \dots, x_p).$$

On the other hand, using that  $x_i = x_{i+1}$ ,

$$\begin{aligned} \tau_i f(x_1, \dots, x_p) &= f(x_{\tau_i(1)}, \dots, x_{\tau_i(i)}, x_{\tau_i(i+1)}, \dots, x_{\tau_i(p)}) \\ &= f(x_1, \dots, x_{i+1}, x_i, \dots, x_p) \\ &= f(x_1, \dots, x_i, x_{i+1}, \dots, x_p). \end{aligned}$$

Hence

$$-f(x_1, \dots, x_p) = f(x_1, \dots, x_p),$$

which implies that  $f(x_1, \dots, x_p) = 0$ . This shows that  $f \in \mathcal{A}_p(E; \mathbb{R})$ .  $\square$

**Theorem 6.** Let  $f \in \mathcal{A}_p(E; \mathbb{R})$ . If  $(x_1, \dots, x_p) \in E^p$  with  $x_i = x_j$  for some  $i \neq j$ , then  $f(x_1, \dots, x_p) = 0$ .

*Proof.* Check that there is some  $\sigma \in S_p$  satisfying  $\sigma(1) = i$  and  $\sigma(2) = j$ . For this  $\sigma$ ,

$$\begin{aligned} (\sigma f)(x_1, \dots, x_p) &= f(x_i, x_j, x_{\sigma(3)}, \dots, x_{\sigma(p)}) \\ &= f(x_i, x_i, x_{\sigma(3)}, \dots, x_{\sigma(p)}) \\ &= 0. \end{aligned}$$

But  $(\sigma f) = (\text{sgn } \sigma)f$ , so  $(\text{sgn } \sigma)f(x_1, \dots, x_p) = 0$ . Therefore  $f(x_1, \dots, x_p) = 0$ .  $\square$

## 2. WEDGE PRODUCTS

A permutation  $\sigma \in S_{p+q}$  is called a  $(p, q)$ -riffle shuffle if

$$\sigma(1) < \dots < \sigma(p), \quad \sigma(p+1) < \dots < \sigma(p+q).$$

Denote by  $S_{p,q}$  those elements of  $S_{p+q}$  that are  $(p, q)$ -riffle shuffles.

**Lemma 7.**  $|S_{p,q}| = \binom{p+q}{p} = \frac{(p+q)!}{p!q!}$ .

Let  $\mathcal{A}_{p,q}(E; \mathbb{R})$  be the set of those  $h \in \mathcal{L}_{p+q}(E; \mathbb{R})$  such that (i) for each  $(y_1, \dots, y_q) \in E^q$ , the map

$$(x_1, \dots, x_p) \mapsto h(x_1, \dots, x_p, y_1, \dots, y_q), \quad E^p \rightarrow \mathbb{R},$$

belongs to  $\mathcal{A}_p(E; \mathbb{R})$ , and (ii) for  $(x_1, \dots, x_p) \in E^p$ , the map

$$(y_1, \dots, y_q) \mapsto h(x_1, \dots, x_p, y_1, \dots, y_q), \quad E^q \rightarrow \mathbb{R},$$

belongs to  $\mathcal{A}_q(E; \mathbb{R})$ .

**Definition 8.** Define  $\phi_{p,q} : \mathcal{A}_{p,q}(E; \mathbb{R}) \rightarrow \mathcal{A}_{p+q}(E; \mathbb{R})$  by

$$\phi_{p,q}(h) = \sum_{\sigma \in S_{p,q}} (\text{sgn } \sigma)(\sigma h), \quad h \in \mathcal{A}_{p,q}(E; \mathbb{R}),$$

which is a linear map  $\mathcal{A}_{p,q}(E; \mathbb{R}) \rightarrow \mathcal{A}_{p+q}(E; \mathbb{R})$ ; we prove that indeed  $\phi_{p,q}(h) \in \mathcal{A}_{p+q}(E; \mathbb{R})$ .

**Theorem 9.**  $\phi_{p,q} : \mathcal{A}_{p,q}(E; \mathbb{R}) \rightarrow \mathcal{A}_{p+q}(E; \mathbb{R})$ .

*Proof.* Let  $h \in \mathcal{A}_{p,q}(E; \mathbb{R})$ , and say  $(x_1, \dots, x_{p+q}) \in E^{p+q}$  with  $x_k = x_{k+1}$  for some  $1 \leq k < p$ .

Let  $A_1$  be those  $\sigma \in S_{p,q}$  such that  $i = \sigma^{-1}(k)$ ,  $j = \sigma^{-1}(k+1) \leq p$ . For  $\sigma \in A_1$ , by Theorem 6,<sup>1</sup>

$$(\sigma h)(x_1, \dots, x_{p+q}) = h(x_{\sigma(1)}, \dots, x_{\sigma(p)}, \dots, x_{\sigma(p+q)}) = 0.$$

Let  $A_2$  be those  $\sigma \in S_{p,q}$  such that  $\sigma^{-1}(k), \sigma^{-1}(k+1) \geq p+1$ . For  $\sigma \in A_2$ , by Theorem 6,

$$(\sigma h)(x_1, \dots, x_{p+q}) = h(x_{\sigma(1)}, \dots, x_{\sigma(p)}, \dots, x_{\sigma(p+q)}) = 0.$$

Thus

$$\sum_{\sigma \in A_1} (\text{sgn } \sigma)(\sigma h)(x_1, \dots, x_{p+q}) = 0$$

<sup>1</sup> $i, j$  are distinct and  $1 \leq i, j \leq p$ ; they need not be adjacent.

and

$$\sum_{\sigma \in A_2} (\operatorname{sgn} \sigma)(\sigma h)(x_1, \dots, x_{p+q}) = 0.$$

Let  $A_3$  be those  $\sigma \in S_{p,q}$  for which  $\sigma^{-1}(k) < p$  and  $\sigma^{-1}(k+1) \geq p+1$  and let  $A_4$  be those  $\sigma \in S_{p,q}$  for which  $\sigma^{-1}(k) \geq p+1$  and  $\sigma^{-1}(k+1) \leq p$ . If  $\sigma \in A_3$  then

$$(\tau_k \sigma)^{-1}(k) = \sigma^{-1} \tau_k^{-1}(k) = \sigma^{-1}(k+1) \geq p+1$$

and

$$(\tau_k \sigma)^{-1}(k+1) = \sigma^{-1} \tau_k^{-1}(k+1) = \sigma^{-1}(k) < p,$$

so  $\tau_k \sigma \in A_4$ . Likewise, if  $\sigma \in A_4$  then  $\tau_k \sigma \in A_3$ . Thus  $A_4 = \tau_k A_3$ . For  $\sigma \in A_3$ , let  $i = \sigma^{-1}(k)$  and  $j = \sigma^{-1}(k+1)$ , for which  $i < p$  and  $j \geq p+1$ . Then, as  $x_k = x_{k+1}$ ,

$$\begin{aligned} & (\operatorname{sgn} \sigma)(\sigma h)(x_1, \dots, x_{p+q}) + (\operatorname{sgn} \tau_k \sigma)(\tau_k \sigma h)(x_1, \dots, x_{p+q}) \\ &= (\operatorname{sgn} \sigma)h(x_{\sigma(1)}, \dots, x_{\sigma(p+q)}) - (\operatorname{sgn} \sigma)h(x_{\tau_k \sigma(1)}, \dots, x_{\tau_k \sigma(p+q)}) \\ &= (\operatorname{sgn} \sigma)(h(x_{\sigma(1)}, \dots, x_{\sigma(p+q)}) - h(x_{\tau_k \sigma(1)}, \dots, x_{\tau_k \sigma(i)}, \dots, x_{\tau_k \sigma(j)}, \dots, x_{\tau_k \sigma(p+q)})) \\ &= (\operatorname{sgn} \sigma)(h(x_{\sigma(1)}, \dots, x_{\sigma(p+q)}) - h(x_{\tau_k \sigma(1)}, \dots, x_{\tau_k(k)}, \dots, x_{\tau_k(k+1)}, \dots, x_{\tau_k \sigma(p+q)})) \\ &= (\operatorname{sgn} \sigma)(h(x_{\sigma(1)}, \dots, x_{\sigma(p+q)}) - h(x_{\sigma(1)}, \dots, x_{k+1}, \dots, x_k, \dots, x_{\sigma(p+q)})) \\ &= (\operatorname{sgn} \sigma)(h(x_{\sigma(1)}, \dots, x_{\sigma(p+q)}) - h(x_{\sigma(1)}, \dots, x_k, \dots, x_{k+1}, \dots, x_{\sigma(p+q)})) \\ &= (\operatorname{sgn} \sigma)(h(x_{\sigma(1)}, \dots, x_{\sigma(p+q)}) - h(x_{\sigma(1)}, \dots, x_{\sigma(p+q)})) \\ &= 0. \end{aligned}$$

Therefore

$$\sum_{\sigma \in A_3 \cup A_4} (\operatorname{sgn} \sigma)(\sigma h)(x_1, \dots, x_{p+q}) = 0.$$

But  $S_{p,q} = A_1 \cup A_2 \cup A_3 \cup A_4$ , so

$$\phi_{p,q}(h)(x_1, \dots, x_{p+q}) = 0.$$

Thus  $\phi_{p,q}(h) \in \mathcal{A}_{p+q}(E; \mathbb{R})$ .  $\square$

Define  $\pi_{\{1, \dots, p+q\}, \{1, \dots, p\}} : E^{p+q} \rightarrow E^p$  by

$$\pi_{\{1, \dots, p+q\}, \{1, \dots, p\}}(x_1, \dots, x_{p+q}) = (x_1, \dots, x_p), \quad (x_1, \dots, x_{p+q}) \in E^{p+q},$$

and define  $\pi_{\{1, \dots, p+q\}, \{p+1, \dots, p+q\}} : E^{p+q} \rightarrow E^q$  by

$$\pi_{\{1, \dots, p+q\}, \{p+1, \dots, p+q\}}(x_1, \dots, x_{p+q}) = (x_{p+1}, \dots, x_{p+q}), \quad (x_1, \dots, x_{p+q}) \in E^{p+q}.$$

**Definition 10.** Define the **wedge product** by

$$\wedge_{p,q} : \mathcal{A}_p(E; \mathbb{R}) \times \mathcal{A}_q(E; \mathbb{R}) \rightarrow \mathcal{A}_{p+q}(E; \mathbb{R})$$

by, for  $f \in \mathcal{A}_p(E; \mathbb{R})$ ,  $g \in \mathcal{A}_q(E; \mathbb{R})$ ,

$$f \wedge_{p,q} g = \phi_{p,q}((f \circ \pi_{\{1, \dots, p+q\}, \{1, \dots, p\}}) \cdot (g \circ \pi_{\{1, \dots, p+q\}, \{p+1, \dots, p+q\}})),$$

i.e., for  $h = (f \circ \pi_{\{1, \dots, p+q\}, \{1, \dots, p\}}) \cdot (g \circ \pi_{\{1, \dots, p+q\}, \{p+1, \dots, p+q\}})$ ,

$$\begin{aligned} (f \wedge_{p,q} g)(x_1, \dots, x_{p+q}) &= \sum_{\sigma \in S_{p,q}} (\text{sgn } \sigma)(\sigma h) \\ &= \sum_{\sigma \in S_{p,q}} (\text{sgn } \sigma) h(x_{\sigma(1)}, \dots, x_{\sigma(p+q)}) \\ &= \sum_{\sigma \in S_{p,q}} (\text{sgn } \sigma) f(x_{\sigma(1)}, \dots, x_{\sigma(p)}) g(x_{\sigma(p+1)}, \dots, x_{\sigma(p+q)}). \end{aligned}$$

**Theorem 11.** For  $f \in \mathcal{A}_p(E; \mathbb{R})$  and  $g \in \mathcal{A}_q(E; \mathbb{R})$ ,

$$g \wedge_{q,p} f = (-1)^{pq} f \wedge_{p,q} g.$$

*Proof.* Define  $\alpha \in S_{p,q}$  by

$$\alpha(i) = q + i, \quad 1 \leq i \leq p, \quad \alpha(p + i) = i, \quad 1 \leq i \leq q.$$

Then<sup>2</sup>

$$\alpha = \prod_{1 \leq i \leq p} \prod_{1 \leq j \leq q} (i + q - j, i + q - j + 1).$$

Thus

$$\text{sgn } \alpha = \prod_{1 \leq i \leq p} \prod_{1 \leq j \leq q} (-1) = (-1)^{pq}.$$

Let  $\tau \in S_{q,p}$ , then for  $1 \leq i \leq p$ ,

$$(\tau\alpha)(i) = \tau(q + i)$$

and for  $1 \leq i \leq q$ ,

$$(\tau\alpha)(p + i) = \tau(i),$$

But  $\tau \in S_{q,p}$  so

$$\tau(1) < \dots < \tau(q), \quad \tau(q + 1) < \dots < \tau(q + p),$$

thus

$$(\tau\alpha)(1) < \dots < (\tau\alpha)(p), \quad (\tau\alpha)(p + 1) < \dots < (\tau\alpha)(p + q),$$

which means that  $\tau\alpha \in S_{p,q}$ . Likewise, if  $\sigma \in S_{p,q}$  then

$$(\sigma\alpha^{-1})(1) = \sigma(q + 1), \dots, (\sigma\alpha^{-1})(q) = \sigma(p + q)$$

and

$$(\sigma\alpha^{-1})(q + 1) = \sigma(1), \dots, (\sigma\alpha^{-1})(q + p) = \sigma(p),$$

and because  $\sigma \in S_{p,q}$  it follows that  $\sigma\alpha^{-1} \in S_{q,p}$ .

<sup>2</sup>For example, take  $p = 3$  and  $q = 2$ . Then

$$\alpha(1) = 3, \alpha(2) = 4, \alpha(3) = 5, \alpha(4) = 1, \alpha(5) = 2.$$

Here

$$\begin{aligned} \prod_{1 \leq i \leq p} \prod_{1 \leq j \leq q} (i + q - j, i + q - j + 1) &= \prod_{1 \leq i \leq 3} \prod_{1 \leq j \leq 2} (i - j + 2, i - j + 3) \\ &= \prod_{1 \leq i \leq 3} (i + 1, i + 2)(i, i + 1) \\ &= (2, 3)(1, 2)(3, 4)(2, 3)(4, 5)(3, 4) \\ &= \alpha. \end{aligned}$$

$$\begin{aligned}
& \text{Hence for } (x_1, \dots, x_{p+q}) \in E^{p+q}, \\
& (g \wedge_{q,p} f)(x_1, \dots, x_{p+q}) \\
&= \sum_{\tau \in S_{q,p}} (\text{sgn } \tau) g(x_{\tau(1)}, \dots, x_{\tau(q)}) f(x_{\tau(q+1)}, \dots, x_{\tau(q+p)}) \\
&= \sum_{\sigma \in S_{p,q}} (\text{sgn } \sigma \alpha^{-1}) g(x_{(\sigma \alpha^{-1})(1)}, \dots, x_{(\sigma \alpha^{-1})(q)}) f(x_{(\sigma \alpha^{-1})(q+1)}, \dots, x_{(\sigma \alpha^{-1})(q+p)}) \\
&= (\text{sgn } \alpha^{-1}) \sum_{\sigma \in S_{p,q}} (\text{sgn } \sigma) g(x_{\sigma(p+1)}, \dots, x_{\sigma(p+q)}) f(x_{\sigma(1)}, \dots, x_{\sigma(p)}) \\
&= (-1)^{pq} \sum_{\sigma \in S_{p,q}} (\text{sgn } \sigma) f(x_{\sigma(1)}, \dots, x_{\sigma(p)}) g(x_{\sigma(p+1)}, \dots, x_{\sigma(p+q)}) \\
&= (-1)^{pq} (f \wedge_{p,q} g)(x_1, \dots, x_{p+q}).
\end{aligned}$$

Thus

$$g \wedge_{q,p} f = (-1)^{pq} f \wedge_{p,q} g.$$

□

Let  $\mathcal{A}_{p,q,r}(E; \mathbb{R})$  be the set of those  $u \in \mathcal{L}_{p+q+r}(E; \mathbb{R})$  such that (i) for each  $(y_1, \dots, y_q, z_1, \dots, z_r) \in E^{q+r}$ , the map

$$(x_1, \dots, x_p) \mapsto u(x_1, \dots, x_p, y_1, \dots, y_q, z_1, \dots, z_r), \quad E^p \rightarrow \mathbb{R},$$

belongs to  $\mathcal{A}_p(E; \mathbb{R})$ , (ii) for  $(x_1, \dots, x_p, z_1, \dots, z_r) \in E^{p+r}$ , the map

$$(y_1, \dots, y_q) \mapsto u(x_1, \dots, x_p, y_1, \dots, y_q, z_1, \dots, z_r), \quad E^q \rightarrow \mathbb{R},$$

belongs to  $\mathcal{A}_q(E; \mathbb{R})$ , and (iii) for  $(x_1, \dots, x_p, y_1, \dots, y_q) \in E^{p+q}$ , the map

$$(z_1, \dots, z_r) \mapsto u(x_1, \dots, x_p, y_1, \dots, y_q, z_1, \dots, z_r), \quad E^r \rightarrow \mathbb{R},$$

belongs to  $\mathcal{A}_r(E; \mathbb{R})$ .

Let  $S_{p,q,\bar{r}}$  be those  $\sigma \in S_{p+q+r}$  such that

$$\sigma(1) < \dots < \sigma(p), \quad \sigma(p+1) < \dots < \sigma(p+q), \quad \sigma(p+q+i) = p+q+i, 1 \leq i \leq r.$$

Let  $S_{\bar{p},q,r}$  be those  $\sigma \in S_{p+q+r}$  such that

$$\sigma(i) = i, 1 \leq i \leq p, \quad \sigma(p+1) < \dots < \sigma(p+q), \quad \sigma(p+q+1) < \dots < \sigma(p+q+r).$$

Let  $S_{p,q,r}$  be those  $\sigma \in S_{p+q+r}$  such that

$$\sigma(1) < \dots < \sigma(p), \quad \sigma(p+1) < \dots < \sigma(p+q), \quad \sigma(p+q+1) < \dots < \sigma(p+q+r).$$

**Lemma 12.**

$$S_{p+q,r} S_{p,q,\bar{r}} = S_{p,q,r}$$

and

$$S_{p,q+r} S_{\bar{p},q,r} = S_{p,q,r}.$$

*Proof.* Let  $\sigma \in S_{p+q,r}$  and  $\tau \in S_{p,q,\bar{r}}$ . Then

$$\sigma(1) < \dots < \sigma(p+q), \quad \sigma(p+q+1) < \dots < \sigma(p+q+r)$$

and

$$\tau(1) < \dots < \tau(p), \quad \tau(p+1) < \dots < \tau(p+q), \quad \tau(p+q+i) = p+q+i, 1 \leq i \leq r.$$

It follows that

$$(\sigma\tau)(1) < \cdots < (\sigma\tau)(p)$$

and

$$(\sigma\tau)(p+1) < \cdots < (\sigma\tau)(p+q)$$

and for  $1 \leq i \leq r$ ,  $(\sigma\tau)(p+q+i) = \sigma(p+q+i)$ , so

$$(\sigma\tau)(p+q+1) < \cdots < \sigma(p+q+r).$$

Thus  $\sigma\tau \in S_{p,q,r}$ . □

Define  $\phi_{p,q,\bar{r}} : \mathcal{A}_{p,q,r}(E; \mathbb{R}) \rightarrow \mathcal{A}_{p+q,r}(E; \mathbb{R})$  by

$$\phi_{p,q,\bar{r}}(u) = \sum_{\sigma \in S_{p,q,\bar{r}}} (\text{sgn } \sigma)(\sigma u), \quad u \in \mathcal{A}_{p,q,r}(E; \mathbb{R})$$

and define  $\phi_{\bar{p},q,r} : \mathcal{A}_{p,q,r}(E; \mathbb{R}) \rightarrow \mathcal{A}_{p,q+r}(E; \mathbb{R})$  by

$$\phi_{\bar{p},q,r}(u) = \sum_{\sigma \in S_{\bar{p},q,r}} (\text{sgn } \sigma)(\sigma u), \quad u \in \mathcal{A}_{p,q,r}(E; \mathbb{R})$$

**Lemma 13.** For  $u \in \mathcal{A}_{p,q,r}(E; \mathbb{R})$ ,

$$(\phi_{p+q,r} \circ \phi_{p,q,\bar{r}})u = \sum_{\rho \in S_{p,q,r}} (\text{sgn } \rho)\rho u$$

and

$$(\phi_{p,q+r} \circ \phi_{\bar{p},q,r})u = \sum_{\rho \in S_{p,q,r}} (\text{sgn } \rho)\rho u,$$

and so

$$\phi_{p+q,r} \circ \phi_{p,q,\bar{r}} = \phi_{p,q+r} \circ \phi_{\bar{p},q,r}.$$

*Proof.* Applying Lemma 12 we get

$$\begin{aligned} (\phi_{p+q,r} \circ \phi_{p,q,\bar{r}})u &= \sum_{\sigma \in S_{p+q,r}} (\text{sgn } \sigma)\sigma\phi_{p,q,\bar{r}}(u) \\ &= \sum_{\sigma \in S_{p+q,r}} (\text{sgn } \sigma)\sigma \sum_{\tau \in S_{p,q,\bar{r}}} (\text{sgn } \tau)(\tau u) \\ &= \sum_{\sigma \in S_{p+q,r}} (\text{sgn } \sigma\tau) \sum_{\tau \in S_{p,q,\bar{r}}} \sigma\tau u \\ &= \sum_{\rho \in S_{p,q,r}} (\text{sgn } \rho)\rho u \end{aligned}$$

and similarly

$$\begin{aligned} (\phi_{p,q+r} \circ \phi_{\bar{p},q,r})u &= \sum_{\sigma \in S_{p,q+r}} (\text{sgn } \sigma)\sigma\phi_{\bar{p},q,r}(u) \\ &= \sum_{\sigma \in S_{p,q+r}} (\text{sgn } \sigma)\sigma \sum_{\tau \in S_{\bar{p},q,r}} (\text{sgn } \tau)(\tau u) \\ &= \sum_{\sigma \in S_{p,q+r}} (\text{sgn } \sigma\tau) \sum_{\tau \in S_{\bar{p},q,r}} \sigma\tau u \\ &= \sum_{\rho \in S_{p,q,r}} (\text{sgn } \rho)\rho u. \end{aligned}$$

Thus

$$(\phi_{p+q,r} \circ \phi_{p,q,\bar{r}})u = (\phi_{p,q+r} \circ \phi_{\bar{p},q,r})u,$$

from which the claim follows.  $\square$

Define  $\pi_{\{1,\dots,p+q+r\},\{1,\dots,p+q\}} : E^{p+q+r} \rightarrow E^{p+q}$  by, for  $(x_1, \dots, x_{p+q+r}) \in E^{p+q+r}$ ,

$$\pi_{\{1,\dots,p+q+r\},\{1,\dots,p+q\}}(x_1, \dots, x_{p+q+r}) = (x_1, \dots, x_{p+q}).$$

**Theorem 14.** *If  $f \in \mathcal{A}_p(E; \mathbb{R})$ ,  $g \in \mathcal{A}_q(E; \mathbb{R})$ , and  $h \in \mathcal{A}_r(E; \mathbb{R})$ , then*

$$(f \wedge_{p,q} g) \wedge_{p+q,r} h = f \wedge_{p,q+r} (g \wedge_{q,r} h).$$

*Proof.* Define  $u \in \mathcal{A}_{p,q,r}(E; \mathbb{R})$  by

$$u(x_1, \dots, x_{p+q+r}) = f(x_1, \dots, x_p)g(x_{p+1}, \dots, x_{p+q})h(x_{p+q+1}, \dots, x_{p+q+r}).$$

On the one hand,

$$\begin{aligned} & (\phi_{p+q,r} \circ \phi_{p,q,\bar{r}})(u) \\ &= \phi_{p+q,r}((f \wedge_{p,q} g) \circ \pi_{\{1,\dots,p+q+r\},\{1,\dots,p+q\}}) \cdot (h \circ \pi_{\{1,\dots,p+q+r\},\{p+q+1,\dots,p+q+r\}}) \\ &= (f \wedge_{p,q} g) \wedge_{p+q,r} h. \end{aligned}$$

On the other hand,

$$\begin{aligned} & (\phi_{p,q+r} \circ \phi_{\bar{p},q,r})(u) \\ &= \phi_{p,q+r}((f \circ \pi_{\{1,\dots,p+q+r\},\{1,\dots,p\}}) \cdot ((g \wedge_{q,r} h) \circ \pi_{\{1,\dots,p+q+r\},\{p+1,\dots,p+q+r\}})) \\ &= f \wedge_{p,q+r} (g \wedge_{q,r} h). \end{aligned}$$

But by Lemma 13,

$$\phi_{p+q,r} \circ \phi_{p,q,\bar{r}} = \phi_{p,q+r} \circ \phi_{\bar{p},q,r},$$

hence

$$(f \wedge_{p,q} g) \wedge_{p+q,r} h = f \wedge_{p,q+r} (g \wedge_{q,r} h).$$

$\square$

### 3. LINEAR FORMS

Let  $E^* = \mathcal{L}_1(E; \mathbb{R})$ , the **dual space of  $E$** , whose elements we call **linear forms**. It is immediate that  $\mathcal{A}_1(E; \mathbb{R}) = \mathcal{L}_1(E; \mathbb{R}) = E^*$ .

**Theorem 15.** *If  $f_1, \dots, f_n \in E^*$  then for  $(x_1, \dots, x_n) \in E^n$ ,*

$$(f_1 \wedge \dots \wedge f_n)(x_1, \dots, x_n) = \sum_{\sigma \in S_n} (\text{sgn } \sigma) f_1(x_{\sigma(1)}) \dots f_n(x_{\sigma(n)}).$$

*Proof.* For  $n = 1$  the claim is immediate. For  $n = 2$ , on the one hand, using the definition of the wedge product,

$$(f_1 \wedge f_2)(x_1, x_2) = \sum_{\sigma \in S_{1,1}} (\text{sgn } \sigma) f_1(x_{\sigma(1)}) f_2(x_{\sigma(2)}),$$

and as  $S_{1,1} = S_2$  the claim is true for  $n = 2$ . Suppose the claim is true for some  $n \geq 2$  and let  $(f_1, \dots, f_n, f_{n+1}) \in E^*$  and  $(x_1, \dots, x_n, x_{n+1}) \in E^{n+1}$ . Then, setting



$u = f_1 \wedge \cdots \wedge f_n \in \mathcal{A}_n(E; \mathbb{R})$ , we have

$$\begin{aligned}
& (f_1 \wedge \cdots \wedge f_n \wedge f_{n+1})(x_1, \dots, x_n, x_{n+1}) \\
&= (u \wedge_{n,1} f_{n+1})(x_1, \dots, x_n, x_{n+1}) \\
&= \sum_{\sigma \in S_{n,1}} (\operatorname{sgn} \sigma) u(x_{\sigma(1)}, \dots, x_{\sigma(n)}) f_{n+1}(x_{\sigma(n+1)}) \\
&= \sum_{\sigma \in S_{n,1}} (\operatorname{sgn} \sigma) \left( \sum_{\tau \in S_n} (\operatorname{sgn} \tau) f_1(x_{\sigma\tau(1)}) \cdots f_n(x_{\sigma\tau(n)}) \right) f_{n+1}(x_{\sigma(n+1)}) \\
&= \sum_{\rho \in S_{n+1}} (\operatorname{sgn} \rho) f_1(x_{\rho(1)}) \cdots f_n(x_{\rho(n)}) f_{n+1}(x_{\rho(n+1)}),
\end{aligned}$$

thus the claim is true for  $n + 1$ .  $\square$

Let  $f_1, \dots, f_n \in E^*$  and  $x_1, \dots, x_n \in E$  and put

$$a_{i,j} = f_i(x_j), \quad 1 \leq i, j \leq n.$$

The Leibniz formula for the determinant tells us

$$\det a = \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) \prod_{i=1}^n a_{i,\sigma(i)} = \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) \prod_{i=1}^n f_i(x_{\sigma(j)}).$$

Then Theorem 15 gives

$$\det(f_i(x_j))_{1 \leq i, j \leq n} = (f_1 \wedge \cdots \wedge f_n)(x_1, \dots, x_n).$$

**Lemma 16.** *If  $f_1, \dots, f_n \in E^*$  are linearly independent then there are  $x_1, \dots, x_n \in E$  such that*

$$f_i(x_j) = \delta_{i,j}, \quad 1 \leq i, j \leq n.$$

**Theorem 17.**  *$f_1, \dots, f_n \in E^*$  are linearly dependent if and only if*

$$f_1 \wedge \cdots \wedge f_n = 0.$$

*Proof.* Suppose  $f_1, \dots, f_n$  are linearly dependent, say, for some  $\lambda_i \in \mathbb{R}$ ,  $i \neq k$ ,

$$f_k = \sum_{i \neq k} \lambda_i f_i.$$

Then, as  $f_i \wedge f_i = 0$ ,

$$f_1 \wedge \cdots \wedge f_n = 0.$$

Suppose that  $f_1, \dots, f_n \in E^*$  are linearly independent. By Lemma 16, there are  $x_1, \dots, x_n \in E$  such that

$$f_i(x_j) = \delta_{i,j}, \quad 1 \leq i, j \leq n.$$

Then  $\det(f_i(x_j)) = 1$ , and hence

$$(f_1 \wedge \cdots \wedge f_n)(x_1, \dots, x_n) = 1,$$

so  $f_1 \wedge \cdots \wedge f_n$  is not identically 0.  $\square$

4.  $E = \mathbb{R}^k$ 

We now take  $E = \mathbb{R}^k$ . For  $1 \leq i \leq k$  define  $\xi_i \in (\mathbb{R}^k)^*$  by

$$\xi_i(x_1, \dots, x_k) = x_i, \quad (x_1, \dots, x_k) \in \mathbb{R}^k.$$

Let  $e_i = (0, \dots, \underbrace{1}_i, \dots, 0) \in \mathbb{R}^k$  for  $1 \leq i \leq k$ , in other words,

$$\xi_i(e_j) = \delta_{i,j}, \quad 1 \leq i, j \leq k.$$

For  $x \in \mathbb{R}^k$ ,

$$x = \sum_{1 \leq i \leq k} \xi_i(x) e_i.$$

**Theorem 18.** (i) If  $f \in \mathcal{L}_p(\mathbb{R}^k; \mathbb{R})$  then for  $(x_1, \dots, x_p) \in (\mathbb{R}^k)^p$ ,

$$f(x_1, \dots, x_p) = \sum_{1 \leq i_1, \dots, i_p \leq k} f(e_{i_1}, \dots, e_{i_p}) \xi_{i_1}(x_1) \cdots \xi_{i_p}(x_p).$$

(ii) If  $f \in \mathcal{A}_p(\mathbb{R}^k; \mathbb{R})$  then

$$f = \sum_{1 \leq i_1 < \dots < i_p \leq k} f(e_{i_1}, \dots, e_{i_p}) \xi_{i_1} \wedge \cdots \wedge \xi_{i_p}.$$

(iii)

$$\dim \mathcal{A}_p(\mathbb{R}^k; \mathbb{R}) = \binom{k}{p}.$$

(iv) If  $f \in \mathcal{A}_k(\mathbb{R}^k; \mathbb{R})$  then

$$f = f(e_1, \dots, e_n) \xi_1 \wedge \cdots \wedge \xi_k.$$

*Proof.* (i) Let  $f \in \mathcal{L}_p(\mathbb{R}^k; \mathbb{R})$ . For  $(x_1, \dots, x_p) \in (\mathbb{R}^k)^p$ , because  $f : (\mathbb{R}^k)^p \rightarrow \mathbb{R}$  is multilinear,

$$\begin{aligned} f(x_1, \dots, x_p) &= f \left( \sum_{1 \leq i_1 \leq k} \xi_{i_1}(x_1) e_{i_1}, \dots, \sum_{1 \leq i_p \leq k} \xi_{i_p}(x_p) e_{i_p} \right) \\ &= \sum_{1 \leq i_1, \dots, i_p \leq k} \xi_{i_1}(x_1) \cdots \xi_{i_p}(x_p) f(e_{i_1}, \dots, e_{i_p}). \end{aligned}$$

(ii) Let  $f \in \mathcal{A}_p(\mathbb{R}^k; \mathbb{R})$ . Then as  $\sigma f = (\text{sgn } \sigma) f$  for  $\sigma \in S_p$ ,

$$f = \frac{1}{p!} \sum_{\sigma \in S_p} (\text{sgn } \sigma) \sigma f,$$

so for  $(x_1, \dots, x_p) \in (\mathbb{R}^k)^p$ , applying Theorem 15,

$$\begin{aligned}
f(x_1, \dots, x_p) &= \frac{1}{p!} \sum_{\sigma \in S_p} (\operatorname{sgn} \sigma) f(x_{\sigma(1)}, \dots, x_{\sigma(p)}) \\
&= \frac{1}{p!} \sum_{\sigma \in S_p} (\operatorname{sgn} \sigma) \sum_{1 \leq i_1, \dots, i_p \leq k} \xi_{i_1}(x_{\sigma(1)}) \cdots \xi_{i_p}(x_{\sigma(p)}) f(e_{i_1}, \dots, e_{i_p}) \\
&= \frac{1}{p!} \sum_{1 \leq i_1, \dots, i_p \leq k} f(e_{i_1}, \dots, e_{i_p}) \sum_{\sigma \in S_p} (\operatorname{sgn} \sigma) \xi_{i_1}(x_{\sigma(1)}) \cdots \xi_{i_p}(x_{\sigma(p)}) \\
&= \frac{1}{p!} \sum_{1 \leq i_1, \dots, i_p \leq k} f(e_{i_1}, \dots, e_{i_p}) (\xi_{i_1} \wedge \cdots \wedge \xi_{i_p})(x_1, \dots, x_p).
\end{aligned}$$

Since  $f$  is alternating,  $i_r = i_s$  for  $r \neq s$  implies  $f(e_{i_1}, \dots, e_{i_p}) = 0$ . Let

$$\mathcal{I}_{p,k} = \{I \subset \{1, \dots, k\} : |I| = p\};$$

For  $I \in \mathcal{I}_{p,k}$ , define  $I_1, \dots, I_p$  by  $I = \{I_1, \dots, I_p\}$  and  $I_1 < \cdots < I_p$ . For a set  $X$ , let  $S_X$  be the group of bijections  $X \rightarrow X$ . Then, applying Theorem 11, as  $|S_I| = p!$ ,

$$\begin{aligned}
f &= \frac{1}{p!} \sum_{I \in \mathcal{I}_{p,k}} \sum_{\tau \in S_I} f(e_{\tau(I_1)}, \dots, e_{\tau(I_p)}) \xi_{\tau(I_1)} \wedge \cdots \wedge \xi_{\tau(I_p)} \\
&= \frac{1}{p!} \sum_{I \in \mathcal{I}_{p,k}} \sum_{\tau \in S_I} (\operatorname{sgn} \tau) f(e_{I_1}, \dots, e_{I_p}) (\operatorname{sgn} \tau) \xi_{I_1} \wedge \cdots \wedge \xi_{I_p} \\
&= \sum_{I \in \mathcal{I}_{p,k}} f(e_{I_1}, \dots, e_{I_p}) \xi_{I_1} \wedge \cdots \wedge \xi_{I_p}.
\end{aligned}$$

proving the claim.

(iii)  $|\mathcal{I}_{p,k}| = \binom{k}{p}$ .

(iv) This follows from (ii) and the fact that  $|\mathcal{I}_{p,k}| = 1$  with  $\mathcal{I}_{p,k} = \{\{1, \dots, k\}\}$ .  $\square$

## REFERENCES

1. Ralph Abraham and Jerrold E. Marsden, *Foundations of mechanics*, second ed., AMS Chelsea Publishing, Providence, RI, 2008.
2. Henri Cartan, *Differential forms*, Hermann, Paris, 1970.

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