

Pell's equation

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1 Side and diagonal numbers

Heath [17, pp. 117–118]:

$$a_1 = 1, \quad d_1 = 1.$$

$$a_n = a_{n-1} + d_{n-1}, \quad d_n = 2a_{n+1} + d_{n+1}.$$

$$\begin{aligned} d_n^2 - 2a_n^2 &= 4a_{n-1}^2 + 4a_{n-1}d_{n-1} + d_{n-1}^2 - 2(a_{n-1}^2 + 2a_{n-1}d_{n-1} + d_{n-1}^2) \\ &= 4a_{n+1}^2 + 4a_{n+1}d_{n+1} + d_{n+1}^2 - 2a_{n-1}^2 - 4a_{n-1}d_{n-1} - 2d_{n-1}^2 \\ &= 2a_{n-1}^2 - d_{n-1}^2 \\ &= -(d_{n-1}^2 - 2a_{n-1}^2). \end{aligned}$$

As $d_1^2 - 2a_1^2 = -1$,

$$d_n^2 - 2a_n^2 = (-1)^n.$$

$$\begin{pmatrix} a_n \\ d_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ d_{n-1} \end{pmatrix} = A \begin{pmatrix} a_{n-1} \\ d_{n-1} \end{pmatrix}.$$

$$\begin{pmatrix} a_{n+1} \\ d_{n+1} \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$A = PDP^{-1}.$$

$$D = \begin{pmatrix} 1 - \sqrt{2} & 0 \\ 0 & 1 + \sqrt{2} \end{pmatrix}, \quad P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}.$$

$$A^n = \begin{pmatrix} \frac{(1-\sqrt{2})^n}{2} + \frac{(1+\sqrt{2})^n}{2} & -\frac{(1-\sqrt{2})^n}{2\sqrt{2}} + \frac{(1+\sqrt{2})^n}{2\sqrt{2}} \\ -\frac{(1-\sqrt{2})^n}{\sqrt{2}} + \frac{(1+\sqrt{2})^n}{\sqrt{2}} & \frac{(1-\sqrt{2})^n}{2} + \frac{(1+\sqrt{2})^n}{2} \end{pmatrix}.$$

$$\begin{pmatrix} a_{n+1} \\ d_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(1-\sqrt{2})^n - \frac{(1-\sqrt{2})^n}{2\sqrt{2}} + \frac{1}{2}(1+\sqrt{2})^n + \frac{(1+\sqrt{2})^n}{2\sqrt{2}} \\ \frac{1}{2}(1-\sqrt{2})^n - \frac{(1-\sqrt{2})^n}{\sqrt{2}} + \frac{1}{2}(1+\sqrt{2})^n + \frac{(1+\sqrt{2})^n}{\sqrt{2}} \end{pmatrix}.$$

Thus

$$\begin{pmatrix} a_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} a_3 \\ d_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}, \quad \begin{pmatrix} a_4 \\ d_4 \end{pmatrix} = \begin{pmatrix} 12 \\ 17 \end{pmatrix}, \quad \begin{pmatrix} a_5 \\ d_5 \end{pmatrix} = \begin{pmatrix} 29 \\ 41 \end{pmatrix}.$$

2 Diophantus

If $x^2 - Ay^2 = 1$ and $y = m(x+1)$ for rational m , then $y^2 = m^2(x^2 + 2x + 1)$, then $x^2 - Am^2(x^2 + 2x + 1) = 1$, then $(Am^2 - 1)x^2 + 2Am^2x + Am^2 + 1 = 0$. Write $(x+1)(px+q) = (Am^2 - 1)x^2 + 2Am^2x + Am^2 + 1$. Then $px^2 + (p+q)x + q = (Am^2 - 1)x^2 + 2Am^2x + Am^2 + 1$. Then $p = Am^2 - 1$, $q = Am^2 + 1$, and so $p+q = Am^2 - 1 + Am^2 + 1 = 2Am^2$. Thus if $x \neq -1$ then $px+q = 0$ and hence $(Am^2 - 1)x + Am^2 + 1 = 0$. Hence $(Am^2 - 1)x = -(Am^2 + 1)$ and so $x = -\frac{Am^2+1}{Am^2-1}$. Thus

$$y = m(x+1) = m \left(-\frac{Am^2+1}{Am^2-1} + 1 \right) \frac{m}{Am^2-1} (-Am^2-1+Am^2-1) = \frac{-2m}{Am^2-1}.$$

Therefore for rational m ,

$$x = -\frac{Am^2+1}{Am^2-1}, \quad y = \frac{-2m}{Am^2-1}$$

satisfy $x^2 - Ay^2 = 1$. cf. Heath [17, pp. 68–69], Nesselmann [24, p. 331].

Diophantus V.11: $30x^2 + 1 = y^2$. Say $y = 5x + 1$. $y^2 = 25x^2 + 10x + 1$. Then $5x^2 - 10x = 0$, so $x = 0$ or $5x - 10 = 0$, i.e. $x = 0$ or $x = 2$. Hence $x = 0, y = 1$ and $x = 2, y = 11$ satisfy $30x^2 + 1 = y^2$.

Diophantus V.14 [17, pp. 211–212]. $34y^2 + 1 = x^2$. Say $x = 6y - 1$. $x^2 = 36y^2 - 12y + 1$. Then $2y^2 - 12y = 0$, i.e. $y(y - 6) = 0$ so $y = 0$ or $y = 6$. Then $x = -1, y = 0$ and $x = 35, y = 6$ satisfy $34y^2 + 1 = x^2$.

3 Fermat

Fermat, February 1657 [31, p. 29]:

Given any number not a square, then there are an infinite number of squares which, when multiplied by the given number, make a square when unity is added.

Example. Given 3, a nonsquare number; this number multiplied by the square number 1, and 1 being added, produces 4, which is a square.

Moreover, the same 3 multiplied by the square 16, with 1 added makes 49, which is a square.

And instead of 1 and 16, an infinite number of squares may be found showing the same property; I demand, however, a general rule, any number being given which is not a square.

It is sought, for example, to find a square which when multiplied into 149, 109, 433, etc., becomes a square when unity is added.

4 Wallis

Wallis [1, p. 546].
Stedall [29]

5 Brouncker

Weil [32, pp. 92–99].

6 Ozanam

Ozanam [25, pp. 503–516], Liv. III, Quest. XXVI.

7 Continued fractions

Let

$$[a_0, a_1] = a_0 + \frac{1}{a_1}$$

and

$$[a_0, \dots, a_{n-1}, a_n] = \left[a_0, \dots, a_{n-2}, a_{n-1} + \frac{1}{a_n} \right].$$

Then for $1 \leq m < n$,

$$[a_0, \dots, a_n] = [a_0, \dots, a_{m-1}, [a_m, \dots, a_n]].$$

Define

$$p_0 = a_0, \quad q_0 = 1, \quad p_1 = a_1 a_0 + 1, \quad q_1 = a_1$$

and for $n \geq 2$,

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2}.$$

Hardy and Wright [13, p. 130, Theorem 149]. For $n \geq 0$,

$$[a_0, \dots, a_n] = \frac{p_n}{q_n}.$$

For $n \geq 1$,

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}.$$

For $n \geq 2$,

$$p_n q_{n-2} - p_{n-2} q_n = (-1)^n a_n.$$

For $n \geq 0$ let

$$a'_n = [a_n, a_{n+1}, \dots].$$

For $x = [a_0, a_1, \dots]$,

$$x = \frac{a'_n p_{n-1} + p_{n-2}}{a'_n q_{n-1} + q_{n-2}}, \quad n \geq 2.$$

Hardy and Wright [13, p. 144, Theorem 176]. A continued fraction $[a_0, a_1, \dots]$ is said to be **periodic** if there is some $L \geq 0$ and some $k \geq 1$ such that $a_{l+k} = a_l$ for all $l \geq L$.

Theorem 1. *If $x = [a_0, a_1, \dots]$ is a periodic continued fraction, then x is a quadratic surd.*

Proof. Let

$$\overline{a_L, \dots, a_{L+k-1}} = [a_L, a_{L+1}, \dots] = a'_L.$$

Thus

$$\begin{aligned} [a_0, \dots, a_{L-1}, a_L, a_{L+1}, \dots] &= [a_0, \dots, a_{L-1}, a'_L] \\ &= [a_0, \dots, a_{L-1}, \overline{a_L, \dots, a_{L+k-1}}]. \end{aligned}$$

As $a_{L+k} = a_L, a_{L+k+1} = a_{L+1}, \dots$,

$$\begin{aligned} a'_L &= [a_L, a_{L+1}, \dots] \\ &= [a_L, a_{L+1}, \dots, a_{L+k-1}, a_L, a_{L+1}, \dots] \\ &= [a_L, a_{L+1}, \dots, a_{L+k-1}, a'_L]. \end{aligned}$$

Let

$$\frac{p'}{q'} = [a_L, a_{L+1}, \dots, a_{L+k-1}], \quad \frac{p''}{q''} = [a_L, a_{L+1}, \dots, a_{L+k-2}].$$

For $t = [a_L, a_{L+1}, \dots, a_{L+k-1}, a'_L]$,

$$t = \frac{a'_L p' + p''}{a'_L q' + q''} = \frac{p' t + p''}{q' t + q''}.$$

Hence $q' t^2 + q'' t = p' t + p''$, so $q' t^2 + (q'' - p') t - p'' = 0$. For $x = [a_0, a_1, \dots]$,

$$x = \frac{a'_L p_{L-1} + p_{L-2}}{a'_L q_{L-1} + q_{L-2}}.$$

Then

$$a'_L = \frac{p_{L-2} - q_{L-2} x}{q_{L-1} x - p_{L-1}}.$$

Thus, with $t = a'_L$,

$$q' \left(\frac{p_{L-2} - q_{L-2} x}{q_{L-1} x - p_{L-1}} \right)^2 + (q'' - p') \frac{p_{L-2} - q_{L-2} x}{q_{L-1} x - p_{L-1}} - p'' = 0.$$

Then

$$q'(p_{L-2}-q_{L-2}x)^2+(q''-p')(p_{L-2}-q_{L-2}x)(q_{L-1}x-p_{L-1})-p''(q_{L-1}x-p_{L-1})^2=0.$$

Therefore there are integers a, b, c such that

$$ax^2+bx+c=0.$$

This means that x is a quadratic surd, as x is irrational. \square

Example. Say $x = [3, 2, 7, 4, \overline{5, 1, 12}]$. $L = 4, k = 3$.

$$\frac{p'}{q'} = [5, 1, 12] = \frac{77}{13}, \quad \frac{p''}{q''} = [5, 1] = \frac{6}{1}.$$

$$\frac{p_{L-1}}{q_{L-1}} = \frac{p_3}{q_3} = [3, 2, 7, 4] = \frac{215}{62}, \quad \frac{p_{L-2}}{q_{L-2}} = \frac{p_2}{q_2} = [3, 2, 7] = \frac{52}{15}.$$

Then

$$\begin{aligned} & q'(p_{L-2}-q_{L-2}x)^2+(q''-p')(p_{L-2}-q_{L-2}x)(q_{L-1}x-p_{L-1})-p''(q_{L-1}x-p_{L-1})^2 \\ &= 13(52-15x)^2+(1-77)(52-15x)(62x-215)-6(62x-215)^2 \\ &= 50541x^2-350444x+607482. \end{aligned}$$

Hence $x = [3, 2, 7, 4, \overline{5, 1, 12}]$ satisfies

$$50541x^2-350444x+607482=0.$$

In fact,

$$x = \frac{175222 + \sqrt{1522}}{50541}.$$

Hardy and Wright [13, p. 144, Theorem 177].

Theorem 2. *If x is a quadratic surd, then the continued fraction of x is periodic.*

Example. Say $x^2 = 218$. $14^2 = 196$.

$$\sqrt{218} = 14 + \sqrt{218} - 14 = 14 + \frac{1}{\frac{1}{\sqrt{218} - 14}}.$$

$$(\sqrt{218} - 14)(\sqrt{218} + 14) = 218 - 196 = 22, \quad \frac{1}{\sqrt{218} - 14} = \frac{\sqrt{218} + 14}{22}.$$

We do not need to compute the decimal expansion of $\sqrt{218}$; we merely have to calculate $\lfloor \frac{\sqrt{218}+14}{22} \rfloor$. Using $14 < \sqrt{218} < 15$,

$$\frac{1}{\sqrt{218} - 14} = 1 + \frac{\sqrt{218} + 14}{22} - 1 = 1 + \frac{\sqrt{218} - 8}{22}.$$

Then

$$\sqrt{218} = 14 + \frac{1}{1 + \frac{\sqrt{218} - 8}{22}} = 14 + \frac{1}{1 + \frac{1}{\frac{22}{\sqrt{218} - 8}}}$$

$$(\sqrt{218} - 8)(\sqrt{218} + 8) = 218 - 64 = 154, \quad \frac{1}{\sqrt{218} - 8} = \frac{\sqrt{218} + 8}{154}.$$

Then

$$\frac{22}{\sqrt{218} - 8} = \frac{22\sqrt{218} + 176}{154}.$$

Using that $14 < \sqrt{218} < 15$,

$$\frac{22}{\sqrt{218} - 8} = 3 + \frac{22\sqrt{218} + 176}{154} - 3 = 3 + \frac{22\sqrt{218} - 286}{154}.$$

Then

$$\sqrt{218} = 14 + \frac{1}{1 + \frac{1}{3 + \frac{22\sqrt{218} - 286}{154}}} = 14 + \frac{1}{1 + \frac{1}{3 + \frac{1}{\frac{154}{22\sqrt{218} - 286}}}}.$$

$$(22\sqrt{218} - 286)(22\sqrt{218} + 286) = 22^2 \cdot 218 - 286^2 = 23716,$$

$$\frac{1}{22\sqrt{218} - 286} = \frac{22\sqrt{218} + 286}{23716}.$$

$$\frac{154}{22\sqrt{218} - 286} = \frac{3388\sqrt{218} + 44044}{23716}.$$

Using $14 < \sqrt{218} < 15$,

$$\frac{154}{22\sqrt{218} - 286} = 3 + \frac{3388\sqrt{218} + 44044}{23716} - 3 = 3 + \frac{3388\sqrt{218} - 27104}{23716}.$$

Then

$$\sqrt{218} = 14 + \frac{1}{1 + \frac{1}{3 + \frac{1}{3 + \frac{3388\sqrt{218} - 27104}{23716}}}} = 14 + \frac{1}{1 + \frac{1}{3 + \frac{1}{3 + \frac{1}{3388\sqrt{218} - 27104}}}}.$$

$$(3388\sqrt{218} - 27104)(3388\sqrt{218} + 27104) = 3388^2 \cdot 218 - 27104^2 = 1767695776,$$

$$\begin{aligned} \frac{1}{3388\sqrt{218} - 27104} &= \frac{3388\sqrt{218} + 27104}{1767695776}. \\ \frac{23716}{3388\sqrt{218} - 27104} &= 23716 \cdot \frac{3388\sqrt{218} + 27104}{1767695776}. \end{aligned}$$

Using $14 < \sqrt{218} < 15$,

$$\begin{aligned} \frac{23716}{3388\sqrt{218} - 27104} &= 1 + 23716 \cdot \frac{3388\sqrt{218} + 27104}{1767695776} - 1 \\ &= 1 + \frac{80349808\sqrt{218} - 1124897312}{1767695776}. \end{aligned}$$

Then

$$\begin{aligned} \sqrt{218} &= 14 + \frac{1}{1 + \frac{1}{3 + \frac{1}{3 + \frac{1}{1 + \frac{80349808\sqrt{218} - 1124897312}{1767695776}}}}} \\ &= 14 + \frac{1}{1 + \frac{1}{3 + \frac{1}{3 + \frac{1}{1 + \frac{1}{1767695776}}}}} \\ &\quad \frac{1}{80349808\sqrt{218} - 1124897312}. \end{aligned}$$

$$(80349808\sqrt{218} - 1124897312)(80349808\sqrt{218} + 1124897312) = 142034016204011008.$$

$$\frac{1767695776}{80349808\sqrt{218} - 1124897312} = 1767695776 \cdot \frac{80349808\sqrt{218} + 1124897312}{142034016204011008}.$$

Using $14 < \sqrt{218} < 15$, the floor of the above quantity is 28. Hence

$$\begin{aligned} \frac{1767695776}{80349808\sqrt{218} - 1124897312} &= 28 + \frac{142034016204011008\sqrt{218} - 1988476226856154112}{142034016204011008} \\ &= 28 + \sqrt{218} - 14. \end{aligned}$$

Then

$$\sqrt{218} = 14 + \frac{1}{1 + \frac{1}{3 + \frac{1}{3 + \frac{1}{1 + \frac{1}{28 + \sqrt{218} - 14}}}}}$$

Thus for $x = \sqrt{218}$,

$$x - 14 = \frac{1}{1 + \frac{1}{3 + \frac{1}{3 + \frac{1}{1 + \frac{1}{28 + x - 14}}}}}$$

Thus for $t = x - 14$,

$$t = \frac{1}{1 + \frac{1}{3 + \frac{1}{3 + \frac{1}{1 + \frac{1}{28 + t}}}}}$$

Therefore $t = [0, \overline{1, 3, 3, 1, 28}]$. Hence $x = 14 + t = [14, \overline{1, 3, 3, 1, 28}]$:

$$\sqrt{218} = [14, \overline{1, 3, 3, 1, 28}].$$

8 Euler

Euler, *Algebra* [10], Part II, Chapter VII.

9 Lagrange

Konen [20, pp. 75–77].

10 Chakravala

Hankel [12, pp. 200–203]

Strachey [30, pp. 36–53]. Dickson [5, pp. 349–350].

Colebrooke [3, pp. 170–184]

Colebrooke [3, pp. 363–372]
Datta and Singh [4, II, pp. 93–99]
Datta and Singh [4, II, pp. 146–161]
Datta and Singh [4, II, pp. 161–172]
Suppose that p_n, q_n are relatively prime and

$$Aq_n^2 + s_n = p_n^2.$$

If d is a common factor of q_n and s_n then $d \mid p_n^2$, so d is a common factor of p_n^2 and q_n^2 , which implies that p_n and q_n have a common factor, a contradiction. Therefore q_n and s_n are relatively prime. Because q_n and s_n are relatively prime, by the Kuttaka algorithm there are some ρ_n, ρ'_n satisfying $-q_n\rho_n + s_n\rho'_n = p_n$. For $r_n = \rho_n + k_n s_n$, $r'_n = \rho'_n + k_n q_n$,

$$\begin{aligned} -q_n r_n + s_n r'_n &= -q_n(\rho_n + k_n s_n) + s_n(\rho'_n + k_n q_n) \\ &= -q_n \rho_n - k_n q_n s_n + s_n \rho'_n + k_n q_n s_n \\ &= -q_n \rho_n + s_n \rho'_n \\ &= p_n. \end{aligned}$$

Take $r_n < \sqrt{A} < r_n + |s_n|$. $r'_n = \frac{p_n + q_n r_n}{s_n}$. Let

$$q_{n+1} = r'_n, \quad p_{n+1} = \frac{p_n q_{n+1} - 1}{q_n}, \quad s_{n+1} = p_{n+1}^2 - A q_{n+1}^2.$$

Example. $69y^2 + 1 = x^2$. $A = 69$.

$Aq_0^2 + s_0 = p_0^2$: $p_0 = 8, q_0 = 1, s_0 = -5$.

$p_0 + q_0 \rho_0 = \rho'_0 s_0$ is equivalent to $8 + \rho_0 = -5\rho'_0$. It is satisfied by $\rho_0 = -8, \rho'_0 = 0$. Take $r_0 = -8 - 5k_0 = 7$. $r'_0 = \frac{p_0 + q_0 r_0}{s_0} = \frac{8 + 1 \cdot 7}{-5} = -3$.

$q_1 = -3$.

$$p_1 = \frac{p_0 q_1 - 1}{q_0} = \frac{8 \cdot -3 - 1}{1} = -25.$$

$$s_1 = p_1^2 - A q_1^2 = 4.$$

$p_1 + q_1 \rho_1 = \rho'_1 s_1$ is equivalent to $-25 - 3\rho_1 = 4\rho'_1$. This is satisfied by $\rho_1 = 1, \rho'_1 = -7$. Take $r_1 = 1 + 4k_1 = 5$. Then $r'_1 = \frac{p_1 + q_1 r_1}{s_1} = \frac{-25 - 3 \cdot 5}{4} = -10$.

$q_2 = -10$.

$$p_2 = \frac{p_1 q_2 - 1}{q_1} = \frac{-25 \cdot -10 - 1}{-3} = -83.$$

$$s_2 = p_2^2 - A q_2^2 = -11.$$

$p_2 + q_2 \rho_2 = \rho'_2 s_2$ is equivalent to $-83 - 10\rho_2 = -11\rho'_2$. This is satisfied by $\rho_2 = 6, \rho'_2 = 13$. Take $r_2 = 6 - 11k_2 = 6$. Then $r'_2 = \frac{p_2 + q_2 r_2}{s_2} = \frac{-83 - 10 \cdot 6}{-11} = 13$.

$q_3 = 13$.

$$p_3 = \frac{p_2 q_3 - 1}{q_2} = \frac{-83 \cdot 13 - 1}{-10} = 108.$$

$$s_3 = p_3^2 - A q_3^2 = 3.$$

$p_3 + q_3\rho_3 = \rho'_3 s_3$ is equivalent to $108 + 13\rho_3 = 3\rho'_3$. This is satisfied by $\rho_3 = 0, \rho'_3 = 36$. Take $r_3 = 36 + 3k_3 = 6$. Then $r'_3 = \frac{p_3+q_3r_3}{s_3} = \frac{108+13\cdot 6}{3} = 62$.
 $q_4 = 62$.

$$p_4 = \frac{p_3q_4 - 1}{q_3} = \frac{108 \cdot 62 - 1}{13} = 515.$$

$$s_4 = p_4^2 - Aq_4^2 = -11.$$

$p_4 + q_4\rho_4 = \rho'_4 s_4$ is equivalent to $515 + 62\rho_4 = -11\rho'_4$. This is satisfied by $\rho_4 = 5, \rho'_4 = -75$. Take $r_4 = 5 - 11k_4 = 5$. Then $r'_4 = \frac{p_4+q_4r_4}{s_4} = \frac{515+62\cdot 5}{-11} = -75$.
 $q_5 = -75$.

$$p_5 = \frac{p_4q_5 - 1}{q_4} = \frac{515 \cdot -75 - 1}{62} = -623.$$

$$s_5 = p_5^2 - Aq_5^2 = 4.$$

$p_5 + q_5\rho_5 = \rho'_5 s_5$ is equivalent to $-623 - 75\rho_5 = 4\rho'_5$. This is satisfied by $\rho_5 = 3, \rho'_5 = -212$. Take $r_5 = 3 + 4k_5 = 7$. Then $r'_5 = \frac{p_5+q_5r_5}{s_5} = \frac{-623-75\cdot 7}{4} = -287$.

$$q_6 = -287.$$

$$p_6 = \frac{p_5q_6 - 1}{q_5} = \frac{-623 \cdot -287 - 1}{-75} = -2384.$$

$$s_6 = p_6^2 - Aq_6^2 = -5.$$

$p_6 + q_6\rho_6 = \rho'_6 s_6$ is equivalent to $-2384 - 287\rho_6 = -5\rho'_6$. This is satisfied by $\rho_6 = 3, \rho'_6 = 649$. Take $r_6 = 3 - 5k_6 = 8$. Then $r'_6 = \frac{p_6+q_6r_6}{s_6} = \frac{-2384-287\cdot 8}{-5} = 936$.
 $q_7 = 936$.

$$p_7 = \frac{p_6q_7 - 1}{q_6} = \frac{-2384 \cdot 936 - 1}{-287} = 7775.$$

$$s_7 = p_7^2 - Aq_7^2 = 1.$$

Therefore

$$7775^2 - 69 \cdot 936^2 = 1.$$

Thus $\sqrt{69} \sim \frac{7775}{936}$.

Example. $91y^2 + 1 = x^2$. $A = 91$.

$$Aq_0^2 + s_0 = p_0^2: p_0 = 10, q_0 = 1, s_0 = 9.$$

$p_0 + q_0\rho_0 = \rho'_0 s_0$ is equivalent to $10 + \rho_0 = 9\rho'_0$. This is satisfied by $\rho_0 = -10, \rho'_0 = 0$. Take $r_0 = -10 + 9k_0 = 8$. Then $r'_0 = \frac{p_0+q_0r_0}{s_0} = \frac{10+1\cdot 8}{9} = 2$.

$$q_1 = 2.$$

$$p_1 = \frac{p_0q_1 - 1}{q_0} = \frac{10 \cdot 2 - 1}{1} = 19.$$

$$s_1 = p_1^2 - Aq_1^2 = -3.$$

$p_1 + q_1\rho_1 = \rho'_1 s_1$ is equivalent with $19 + 2\rho_1 = -3\rho'_1$. This is satisfied by $\rho_1 = 1, \rho'_1 = -7$. Take $r_1 = 1 - 3k_1 = 7$. Then $r'_1 = \frac{p_1+q_1r_1}{s_1} = \frac{19+2\cdot 7}{-3} = -11$.

$$q_2 = -11.$$

$$p_2 = \frac{p_1q_2 - 1}{q_1} = \frac{19 \cdot -11 - 1}{2} = -105.$$

$$s_2 = p_2^2 - Aq_2^2 = 14.$$

$p_2 + q_2\rho_2 = \rho_2's_2$ is equivalent with $-105 - 11\rho_2 = 14\rho_2'$. This is satisfied by $\rho_2 = 7, \rho_2' = -13$. Take $r_2 = 7, r_2' = -13$.

$$q_3 = -13.$$

$$p_3 = \frac{p_2q_3 - 1}{q_2} = \frac{-105 \cdot -13 - 1}{-11} = -124.$$

$$s_3 = p_3^2 - Aq_3^2 = -3.$$

$p_3 + q_3\rho_3 = \rho_3's_3$ is equivalent with $-124 - 13\rho_3 = -3\rho_3'$. This is satisfied by $\rho_3 = 2, \rho_3' = 50$. Take $r_3 = 2 - 3k_3 = 8$. Then $r_3' = \frac{p_3+q_3r_3}{s_3} = \frac{-124-13 \cdot 8}{-3} = 76$.

$$q_4 = 76.$$

$$p_4 = \frac{p_3q_4 - 1}{q_3} = \frac{-124 \cdot 76 - 1}{-13} = 725.$$

$$s_4 = p_4^2 - Aq_4^2 = 9.$$

$p_4 + q_4\rho_4 = \rho_4's_4$ is equivalent with $725 + 76\rho_4 = 9\rho_4'$. This is satisfied by $\rho_4 = 1, \rho_4' = 89$. Take $r_4 = 1, r_4' = 89$.

$$q_5 = 89.$$

$$p_5 = \frac{p_4q_5 - 1}{q_4} = \frac{725 \cdot 89 - 1}{76} = 849.$$

$$s_5 = p_5^2 - Aq_5^2 = -10.$$

$p_5 + q_5\rho_5 = \rho_5's_5$ is equivalent with $849 + 89\rho_5 = -10\rho_5'$. This is satisfied by $\rho_5 = 9, \rho_5' = -165$. Take $r_5 = 9, r_5' = -165$.

$$q_6 = -165.$$

$$p_6 = \frac{p_5q_6 - 1}{q_5} = \frac{849 \cdot -165 - 1}{89} = -1574.$$

$$s_6 = p_6^2 - Aq_6^2 = 1.$$

Therefore

$$1574^2 - 91 \cdot 165^2 = 1.$$

Thus $\sqrt{91} \sim \frac{1574}{165}$.

Example. $109y^2 + 1 = x^2$. $A = 109$.

$$Ay_0^2 + s_0 = x_0^2: x_0 = 10, y_0 = 1, s_0 = -9.$$

$x_0 + y_0\rho_0 = s_0\rho_0'$ is equivalent to $10 + \rho_0 = -9\rho_0'$. This is satisfied by $\rho_0 = -10, \rho_0' = 0$. Take $r_0 = -10 + 9k_0 = 8$. Then $r_0' = \frac{x_0+y_0r_0}{s_0} = \frac{10+1 \cdot 8}{-9} = -2$.

$$y_1 = -2.$$

$$x_1 = \frac{x_0y_1 - 1}{y_0} = \frac{10 \cdot -2 - 1}{1} = -21.$$

$$s_1 = x_1^2 - Ay_1^2 = 5.$$

$x_1 + y_1\rho_1 = \rho_1's_1$ is equivalent with $-21 - 2\rho_1 = 5\rho_1'$. This is satisfied by $\rho_1 = 2, \rho_1' = -5$. Take $r_1 = 2 + 5k_1 = 7$. Then $r_1' = \frac{x_1+y_1r_1}{s_1} = \frac{-21-2 \cdot 7}{5} = -7$.

$$y_2 = -7.$$

$$x_2 = \frac{x_1 y_2 - 1}{y_1} = \frac{-21 \cdot -7 - 1}{-2} = -73.$$

$$s_2 = x_2^2 - A y_2^2 = -12.$$

$x_2 + y_2 \rho_2 = \rho_2' s_2$ is equivalent with $-73 - 7\rho_2 = -12\rho_2'$. This is satisfied by $\rho_2 = 5, \rho_2' = 9$. Take $r_2 = 5, r_2' = 9$.

$$y_3 = 9.$$

$$x_3 = \frac{x_2 y_3 - 1}{y_2} = \frac{-73 \cdot 9 - 1}{-7} = 94.$$

$$s_3 = x_3^2 - A y_3^2 = 7.$$

$x_3 + y_3 \rho_3 = \rho_3' s_3$ is equivalent with $94 + 9\rho_3 = 7\rho_3'$. This is satisfied by $\rho_3 = 2, \rho_3' = 16$. Take $r_3 = 2 + 7k_3 = 9$. Then $r_3' = \frac{x_3 + y_3 r_3}{s_3} = \frac{94 + 9 \cdot 9}{7} = 25$.

$$y_4 = 25.$$

$$x_4 = \frac{x_3 y_4 - 1}{y_3} = \frac{94 \cdot 25 - 1}{9} = 261.$$

$$s_4 = x_4^2 - A y_4^2 = -4.$$

$x_4 + y_4 \rho_4 = \rho_4' s_4$ is equivalent with $261 + 25\rho_4 = -4\rho_4'$. This is satisfied by $\rho_4 = 3, \rho_4' = -84$. Take $r_4 = 3 - 4k_4 = 7$. Then $r_4' = \frac{x_4 + y_4 r_4}{s_4} = \frac{261 + 25 \cdot 7}{-4} = -109$.

$$y_5 = -109.$$

$$x_5 = \frac{x_4 y_5 - 1}{y_4} = \frac{261 \cdot -109 - 1}{25} = -1138.$$

$$s_5 = x_5^2 - A y_5^2 = 15.$$

$x_5 + y_5 \rho_5 = \rho_5' s_5$ is equivalent with $-1138 - 109\rho_5 = 15\rho_5'$. This is satisfied by $\rho_5 = 8, \rho_5' = -134$. Take $r_5 = 8, r_5' = -134$.

$$y_6 = -134.$$

$$x_6 = \frac{x_5 y_6 - 1}{y_5} = \frac{-1138 \cdot -134 - 1}{-109} = -1399.$$

$$s_6 = x_6^2 - A y_6^2 = -3.$$

$x_6 + y_6 \rho_6 = \rho_6' s_6$ is equivalent with $-1399 - 134\rho_6 = -3\rho_6'$. This is satisfied by $\rho_6 = 1, \rho_6' = 511$. Take $r_6 = 1 - 3k_6 = 10$. Then $r_6' = \frac{x_6 + y_6 r_6}{s_6} = \frac{-1399 - 134 \cdot 10}{-3} = 913$.

$$y_7 = 913.$$

$$x_7 = \frac{x_6 y_7 - 1}{y_6} = \frac{-1399 \cdot 913 - 1}{-134} = 9532.$$

$$s_7 = x_7^2 - A y_7^2 = 3.$$

$x_7 + y_7 \rho_7 = \rho_7' s_7$ is equivalent with $9532 + 913\rho_7 = 3\rho_7'$. This is satisfied by $\rho_7 = 2, \rho_7' = 3786$. Take $r_7 = 2 + 3k_7 = 8$. Then $r_7' = \frac{x_7 + y_7 r_7}{s_7} = \frac{9532 + 913 \cdot 8}{3} = 5612$.

$$y_8 = 5612.$$

$$x_8 = \frac{x_7 y_8 - 1}{y_7} = \frac{9532 \cdot 5612 - 1}{913} = 58591.$$

$$s_8 = x_8^2 - A y_8^2 = -15.$$

$x_8 + y_8 \rho_8 = \rho'_8 s_8$ is equivalent with $58591 + 5612 \rho_8 = -15 \rho'_8$. This is satisfied by $\rho_8 = 7, \rho'_8 = -6525$. Take $r_8 = 7, r'_8 = -6525$.

$$y_9 = -6525.$$

$$x_9 = \frac{x_8 y_9 - 1}{y_8} = \frac{58591 \cdot -6525 - 1}{5612} = -68123.$$

$$s_9 = x_9^2 - A y_9^2 = 4.$$

$x_9 + y_9 \rho_9 = \rho'_9 s_9$ is equivalent with $-68123 - 6525 \rho_9 = 4 \rho'_9$. This is satisfied by $\rho_9 = 1, \rho'_9 = -18662$. Take $r_9 = 1 + 4k_9 = 9$. Then $r'_9 = \frac{x_9 + y_9 r_9}{s_9} = \frac{-68123 - 6525 \cdot 9}{4} = -31712$.

$$y_{10} = -31712.$$

$$x_{10} = \frac{x_9 y_{10} - 1}{y_9} = \frac{-68123 \cdot (-31712) - 1}{-6525} = -331083.$$

$$s_{10} = x_{10}^2 - A y_{10}^2 = -7.$$

$x_{10} + y_{10} \rho_{10} = \rho'_{10} s_{10}$ is equivalent with $-331083 - 31712 \rho_{10} = -7 \rho'_{10}$. This is satisfied by $\rho_{10} = 5, \rho'_{10} = 69949$. Take $r_{10} = 5, r'_{10} = 69949$.

$$y_{11} = 69949.$$

$$x_{11} = \frac{x_{10} y_{11} - 1}{y_{10}} = \frac{-331083 \cdot 69949 - 1}{-31712} = 730289.$$

$$s_{11} = x_{11}^2 - A y_{11}^2 = 12.$$

$x_{11} + y_{11} \rho_{11} = \rho'_{11} s_{11}$ is equivalent with $730289 + 69949 \rho_{11} = 12 \rho'_{11}$. This is satisfied by $\rho_{11} = 7, \rho'_{11} = 101661$. Take $r_{11} = 5, r'_{10} = 101661$.

$$y_{12} = 101661.$$

$$x_{12} = \frac{x_{11} y_{12} - 1}{y_{11}} = \frac{730289 \cdot 101661 - 1}{69949} = 1061372.$$

$$s_{12} = x_{12}^2 - A y_{12}^2 = -5.$$

$x_{12} + y_{12} \rho_{12} = \rho'_{12} s_{12}$ is equivalent with $1061372 + 101661 \rho_{12} = -5 \rho'_{12}$. This is satisfied by $\rho_{12} = 3, \rho'_{12} = -273271$. Take $r_{12} = 3 - 5k_{12} = 8$. Then $r'_{12} = \frac{x_{12} + y_{12} r_{12}}{s_{12}} = \frac{1061372 + 101661 \cdot 8}{-5} = -374932$.

$$y_{13} = -374932.$$

$$x_{13} = \frac{x_{12} y_{13} - 1}{y_{12}} = \frac{1061372 \cdot (-374932) - 1}{101661} = -3914405.$$

$$s_{13} = x_{13}^2 - Ay_{13}^2 = 9.$$

$x_{13} + y_{13}\rho_{13} = \rho'_{13}s_{13}$ is equivalent with $-3914405 - 374932\rho_{13} = 9\rho'_{13}$. This is satisfied by $\rho_{13} = 1, \rho'_{13} = -476593$. Take $r_{13} = 1 + 9k_{13} = 10$. Then $r'_{13} = \frac{x_{13} + y_{13}r_{13}}{s_{13}} = \frac{-3914405 - 374932 \cdot 10}{9} = -851525$.
 $y_{14} = -851525$.

$$x_{14} = \frac{x_{13}y_{14} - 1}{y_{13}} = \frac{-3914405 \cdot (-851525) - 1}{-374932} = -8890182.$$

$$s_{14} = x_{14}^2 - Ay_{14}^2 = -1.$$

$x_{14} + y_{14}\rho_{14} = \rho'_{14}s_{14}$ is equivalent with $-8890182 - 851525\rho_{14} = -\rho'_{14}$. This is satisfied by $\rho_{14} = 0, \rho'_{14} = 8890182$. Take $r_{14} = -k_{14} = 10$. Then $r'_{14} = \frac{x_{14} + y_{14}r_{14}}{s_{14}} = \frac{-8890182 - 851525 \cdot 10}{-1} = 17405432$.
 $y_{15} = 17405432$.

$$x_{15} = \frac{x_{14}y_{15} - 1}{y_{14}} = \frac{-8890182 \cdot 17405432 - 1}{-851525} = 181718045.$$

$$s_{15} = x_{15}^2 - Ay_{15}^2 = 9.$$

$r_{15} = 8, r'_{15} = 35662389$.
 $y_{16} = 35662389$.

$$x_{16} = \frac{x_{15}y_{16} - 1}{y_{15}} = \frac{35662389 \cdot 35662389 - 1}{17405432} = 372326272.$$

$$s_{16} = x_{16}^2 - Ay_{16}^2 = -5.$$

$\rho_{16} = 2, \rho'_{16} = -88730210$. $r_{16} = 2 - 5k_{16} = 7$. Then $r'_{16} = -124392599$.
 $y_{17} = -124392599$.

$$x_{17} = -1298696861.$$

$$s_{17} = 12.$$

$r_{18} = 5, r'_{18} = -160054988$.
 $y_{18} = -160054988$.

$$x_{18} = -1671023133.$$

$$s_{18} = -7.$$

$\rho_{19} = 2, \rho'_{19} = 284447587$. Take $r_{19} = 2 - 7k_{19} = 9$. Then $r'_{19} = 444502575$.
 $y_{19} = 444502575$.

$$x_{19} = 4640743127.$$

$$s_{19} = 4.$$

$\rho_{20} = 3, \rho'_{20} = 1493562713$. Take $r_{20} = 3 + 4k_{20} = 7$. Then $r'_{20} = 1938065288$.

$y_{20} = 1938065288$.

$$x_{20} = 20233995641.$$

$$s_{20} = -15.$$

$$r_{21} = 8, r'_{21} = -2382567863.$$

$$y_{21} = -2382567863.$$

$$x_{21} = -24874738768.$$

$$s_{21} = 3.$$

$$\rho_{22} = 1, \rho'_{22} = -9085768877. \text{ Take } r_{22} = 1 + 3k_{22} = 10. \text{ Then } r'_{22} = -16233472466.$$

$$y_{22} = -16233472466.$$

$$x_{22} = . - 169482428249.$$

$$s_{22} = -3.$$

$$\rho_{23} = 2, \rho'_{23} = 67316457727. \text{ Take } r_{23} = 2 - 3k_{23} = 8. \text{ Then } r'_{23} = 99783402659.$$

$$y_{23} = 99783402659.$$

$$x_{23} = 1041769308262.$$

$$s_{23} = 15.$$

$$r_{24} = 7, r'_{24} = 116016875125.$$

$$y_{24} = 116016875125.$$

$$x_{24} = 1211251736511.$$

$$s_{24} = -4.$$

$$\rho_{25} = 1, \rho'_{25} = -331817152909. \text{ Take } r_{25} = 1 - 4k_{25} = 9. \text{ Then } r'_{25} = -563850903159.$$

$$y_{25} = -563850903159.$$

$$x_{25} = . - 5886776254306.$$

$$s_{25} = 7.$$

$$r_{26} = 5, r'_{26} = -1243718681443.$$

$$y_{26} = -1243718681443.$$

$$x_{26} = -12984804245123.$$

$$s_{26} = . - 12.$$

$$r_{27} = 7, r'_{27} = 1807569584602.$$

$$y_{27} = 1807569584602.$$

$$x_{27} = 18871580499429.$$

$$s_{27} = 5.$$

$$\rho_{28} = 3, \rho'_{28} = 4858857850647. \text{ Take } r_{28} = 3 + 5k_{28} = 8. \text{ Then } r'_{28} = 6666427435249.$$

$$y_{28} = 6666427435249.$$

$$x_{28} = 69599545743410.$$

$$s_{28} = -9.$$

$$\rho_{29} = 1, \rho'_{29} = -8473997019851. \text{ Take } r_{29} = 1 + 9k_{29} = 10. \text{ Then } r'_{29} = -15140424455100.$$

$$y_{29} = -15140424455100.$$

$$x_{29} = -158070671986249.$$

$$s_{29} = 1.$$

Therefore

$$158070671986249^2 - 109 \cdot 15140424455100^2 = 1.$$

$$\text{Thus } \sqrt{109} \sim \frac{158070671986249}{15140424455100}.$$

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