Notes on the history of Liouville's theorem

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1 Introduction

We denote by $\mathscr{B}(\mathbb{R}^n)$ the set of all linear maps $\mathbb{R}^n \to \mathbb{R}^n$. We take it as known that with the operator norm

$$\|A\| = \sup\{\|Av\| : v \in \mathbb{R}^n, \|v\| \leq 1\}, \qquad A \in \mathscr{B}(\mathbb{R}^n),$$

$\mathscr{B}(\mathbb{R}^n)$ is a Banach space.

2 Autonomous differential equations

Lemma 1. *If $A \in \mathscr{B}(\mathbb{R}^n)$, then*

$$\det(I + \epsilon A + o(\epsilon)) = 1 + \epsilon trA + o(\epsilon)$$

as $\epsilon \to 0$.

Proof. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A, repeated according to algebraic multiplicity. For $\epsilon > 0$, the eigenvalues of $I + \epsilon A + o(\epsilon)$ repeated according to algebraic multiplicity are

$$1 + \epsilon\lambda_1 + o(\epsilon), \ldots, 1 + \epsilon\lambda_n + o(\epsilon),$$

as $\epsilon \to 0$. The determinant of a linear map $\mathbb{R}^n \to \mathbb{R}^n$ is the product of its eigenvalues according to algebraic multiplicity, so

$$\det(I + \epsilon A + o(\epsilon)) = \prod_{k=1}^{n} (1 + \epsilon\lambda_k + o(\epsilon)),$$

as $\epsilon \to 0$. But

$$\prod_{k=1}^{n} (1 + \epsilon\lambda_k + o(\epsilon)) = 1 + \epsilon \sum_{k=1}^{n} \lambda_k + o(\epsilon) = 1 + \epsilon \mathrm{tr} A + o(\epsilon)$$

as $\epsilon \to 0$. □

Theorem 2. *If* $A \in \mathscr{B}(\mathbb{R}^n)$*, then*

$$\det e^A = e^{trA}$$

Proof. We have

$$e^A = \lim_{m\to\infty} \left(I + \frac{A}{m}\right)^m.$$

As $\det : \mathscr{B}(\mathbb{R}^n) \to \mathbb{R}$ is continuous, we have

$$\det e^A = \lim_{m\to\infty} \det \left(I + \frac{A}{m}\right)^m.$$

Then, using Lemma 1,

$$\begin{aligned} \det e^A &= \lim_{m\to\infty} \left(\det \left(I + \frac{A}{m}\right)\right)^m \\ &= \lim_{m\to\infty} \left(1 + \frac{1}{m}\mathrm{tr}A + o\left(\frac{1}{m}\right)\right)^m \\ &= e^{\mathrm{tr}A}. \end{aligned}$$

□

If $A \in \mathscr{B}(\mathbb{R}^n)$, then the flow of the vector field A is

$$(t, x) \mapsto e^{tA}x.$$

For each t we have $e^{tA} \in \mathscr{B}(\mathbb{R}^n)$, and by Theorem 2 we have

$$\det(e^{tA}) = e^{\mathrm{tr}(tA)} = e^{t\mathrm{tr}A}.$$

Let λ be Lebesgue measure on $\mathbb{R}^n$. If U is an open subset of $\mathbb{R}^n$, then

$$\lambda(e^{tA}U) = \int_{e^{tA}U} dy = \int_U |\det(De^{tA})(x)|dx = \int_U |\det(e^{tA})|dx = e^{t\mathrm{tr}A}\lambda(U).$$

Therefore, λ is an invariant measure for the flow if and only if $\mathrm{tr}A = 0$, namely, if and only if A is skew-symmetric.

3 Nonautonomous differential equations

Suppose that I is an open interval and $A \in C(I, \mathscr{B}(\mathbb{R}^n))$. The set X of all functions $x : I \to \mathbb{R}^n$ that satisfy the differential equation

$$\dot{x}(t) = A(t)x(t)$$

is a vector space. For each $t \in I$ we define $B_t : X \to \mathbb{R}^n$ by $B_t(x) = x(t)$. It is apparent that for each $t \in I$ the map B_t is linear. For each $x_0 \in \mathbb{R}^n$, by the existence and uniqueness theorem for ordinary differential equations there is a unique $x \in X$ for which $B_0(x) = x_0$, hence for each $t \in I$ we get that B_t is a bijection, and hence a linear isomorphism.

Suppose that $\phi_1, \ldots, \phi_n \in X$, and for each $t \in I$ let $\Phi(t) \in \mathscr{B}(\mathbb{R}^n)$ be defined by $\Phi(t)e_i = \phi_i(t)$. Then

$$\dot{\Phi}(t)e_i = \frac{d}{dt}(\Phi(t)e_i) = \dot{\phi}_i(t) = A(t)\phi_i(t), \qquad t \in I,$$

and hence

$$\dot{\Phi}(t) = A(t)\Phi(t), \qquad t \in I. \tag{1}$$

The *Wronskian* $W = W(\phi_1, \ldots \phi_n)$ of the ordered set $\phi_1, \ldots, \phi_n$ is the function that assigns to each $t \in I$ the oriented volume of the parallelepiped spanned by $\phi_1(t), \ldots, \phi_n(t)$. That is,

$$W(t) = \det \Phi(t), \qquad t \in I.$$

Theorem 3. *Suppose that I is an open interval and that $A \in C(I, \mathscr{B}(\mathbb{R}^n))$. If $\phi_1, \ldots, \phi_n \in X$, then the Wronskian $W = W(\phi_1, \ldots, \phi_n)$ satisfies*

$$\dot{W}(t) = (trA(t))W(t), \qquad t \in I.$$

Proof. By (1), for each $t \in I$ we have

$$\begin{aligned}
\Phi(t+\Delta) &= \Phi(t) + \dot{\Phi}(t)\Delta + o(\Delta) \\
&= \Phi(t) + A(t)\Phi(t)\Delta + o(\Delta) \\
&= \Phi(t) + A(t)\Phi(t)\Delta + o(\Phi(t)\Delta) \\
&= \Phi(t)(I + A(t)\Delta + o(\Delta)) \\
&= \Phi(t)(I + A(t)\Delta + o(\Delta))
\end{aligned}$$

as $\Delta \to 0$. Using Lemma 1 we get

$$\begin{aligned}
W(t+\Delta) &= \det \Phi(t+\Delta) \\
&= \det \Phi(t) \det(I + A(t)\Delta + o(\Delta)) \\
&= \det \Phi(t)(1 + \mathrm{tr}A(t)\Delta + o(\Delta)) \\
&= \det \Phi(t) + \det \Phi(t)\mathrm{tr}A(t)\Delta + o\left(\det \Phi(t)\Delta\right) \\
&= \det \Phi(t) + \det \Phi(t)\mathrm{tr}A(t)\Delta + o(\Delta)
\end{aligned}$$

as $\Delta \to 0$, i.e.,

$$W(t+\Delta) = W(t) + W(t)\mathrm{tr}A(t)\Delta + o(\Delta),$$

which gives us

$$\dot{W}(t) = (\mathrm{tr}A(t))W(t).$$

□

One checks that for any $t_0 \in I$,

$$t \mapsto W(t_0) \exp\left(\int_{t_0}^t \mathrm{tr} A(\tau) d\tau\right), \qquad t \in I$$

is a solution of the differential equation in Theorem 3. For each $t \in I$ we have that $v \mapsto (\mathrm{tr} A(t))v$ is linear, and in particular is locally Lipschitz, so by the existence and uniqueness theorem it follows that

$$W(t) = W(t_0) \exp\left(\int_{t_0}^t \mathrm{tr} A(\tau) d\tau\right), \qquad t \in I.$$

4 Jacobi's formula

Let Ω be the volume form on $\mathbb{R}^n$, and let $A \in \mathscr{B}(\mathbb{R}^n)$. One checks that

$$\Omega(x_1, \dots, x_n) \det A = \Omega(Ax_1, \dots, Ax_n), \qquad x_1, \dots, x_n \in \mathbb{R}^n. \tag{2}$$

If ω is an $(n-1)$-form on $\mathbb{R}^n$, then there is a unique $x_\omega \in \mathbb{R}^n$ such that for all $x_1, \dots, x_{n-1} \in \mathbb{R}^n$,

$$\omega(x_1, \dots, x_{n-1}) = \Omega(x_\omega, x_1, \dots, x_{n-1}).$$

Thus, if $x_0 \in \mathbb{R}^n$ and we define an $(n-1)$-form ω by

$$\omega(x_1, \dots, x_{n-1}) = \Omega(x_0, Ax_1, \dots, Ax_n),$$

then there is some x_ω with which

$$\Omega(x_0, Ax_1, \dots, Ax_n) = \Omega(x_\omega, x_1, \dots, x_{n-1}).$$

We define $\mathrm{adj}\, A \in \mathscr{B}(\mathbb{R}^n)$ by $(\mathrm{adj}\, A)(x_0) = x_\omega$. Thus, for $x_1, \dots, x_n \in \mathbb{R}^n$, we have

$$\Omega(x_1, Ax_2, \dots, Ax_n) = \Omega((\mathrm{adj}\, A)x_1, x_2, \dots, x_n). \tag{3}$$

Hence

$$\Omega(Ax_1, Ax_2, \dots, Ax_n) = \Omega((\mathrm{adj}\, A)Ax_1, x_2, \dots, x_n),$$

therefore

$$\Omega((\mathrm{adj}\, A)Ax_1, x_2, \dots, x_n) = \Omega(x_1, \dots, x_n) \det A,$$

and because this holds for all $x_1, \dots, x_n \in \mathbb{R}^n$, it follows that

$$(\mathrm{adj}\, A)A = (\det A)I.$$

Furthermore, if $A \in \mathscr{B}(\mathbb{R}^n)$, then one checks that for all $x_1, \dots, x_n \in \mathbb{R}^n$,

$$\Omega(x_1, \dots, x_n)\mathrm{tr} A = \sum_{i=1}^n \Omega(x_1, \dots, Ax_i, \dots, x_n). \tag{4}$$

If I is an open interval and $A \in C^1(I, \mathscr{B}(\mathbb{R}^n))$, we have, using (2) for the first equality, (3) for the third equality, and (4) for the fourth equality,

$$\begin{aligned}
\frac{d}{dt}\Big(\Omega(e_1,\ldots,e_n)\det A(t)\Big) &= \frac{d}{dt}\Omega(A(t)e_1,\ldots,A(t)e_n) \\
&= \sum_{i=1}^n \Omega(A(t)e_1,\ldots,\dot{A}(t)e_i,\ldots,A(t)e_n) \\
&= \sum_{i=1}^n \Omega(e_1,\ldots,(\operatorname{adj} A(t))\dot{A}(t)e_i,\ldots,e_n) \\
&= \Omega(e_1,\ldots,e_n)\operatorname{tr}((\operatorname{adj} A(t))\dot{A}(t)),
\end{aligned}$$

that is,

$$\frac{d}{dt}\det A(t) = \operatorname{tr}((\operatorname{adj} A(t))\dot{A}(t)).$$

Kline p. 798, Jacobi [40], [41, §17], Felix Klein, *19th century*, chapter V

5 Reynolds transport theorem

If V is a vector field with flow ϕ and U is a bounded open subset of $\mathbb{R}^n$ with piecewise smooth boundary and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is smooth, then with $U_t = \phi_t(U)$,

$$\int_{U_t} f(y,t)dy = \int_U f(\phi_t(x),t)\det(D\phi_t)(x)dx;$$

this presumes that $\det(D\phi_t)(x) > 0$. Write $\frac{D}{Dt} = \frac{\partial}{\partial t} + V \cdot D$. We then have

$$\begin{aligned}
\frac{d}{dt}\int_{U_t} f(y,t)dy &= \frac{d}{dt}\int_U f(\phi_t(x),t)\det(D\phi_t)(x)dx \\
&= \int_U (Df)(\phi_t(x),t)\dot{\phi}_t(x)\det(D\phi_t)(x) \\
&\quad + \frac{\partial f}{\partial t}(\phi_t(x),t)\det(D\phi_t)(x) + f(\phi_t(x),t)\frac{d}{dt}\det(D\phi_t)(x)dx \\
&= \int_U \frac{Df}{Dt}(\phi_t(x),t)\det(D\phi_t)(x) + f(\phi_t(x),t)\frac{d}{dt}\det(D\phi_t)(x)dx.
\end{aligned}$$

Writing $J_t(x) = \det(D\phi_t)(x)$, we have

$$\begin{aligned}
\frac{d}{dt}\int_{U_t} f(y,t)dy &= \int_U \frac{Df}{Dt}(\phi_t(x),t)J_t(x) + f(\phi_t(x),t)\frac{d}{dt}J_t(x)dx \\
&= \int_U \frac{D(fJ)}{Dt}
\end{aligned}$$

Reynolds [73, pp. 12–13, art. 14]

Amann and Escher [4, p. 425, Theorem 2.11]

6 Symplectic geometry

7 Geodesic flow

Invariance of a kinematic measure on the unit tangent bundle.

8 References

Jacobi [42, p. 93]
Truesdell [87, pp. 101, 105, 351]
Whittaker [90, p. 323, §148]
Hartman [35, p. 91]
Barrow-Green [6, p. 83]
Goroff [70, p. I79]
Gray [32, p. 380]
Ostrogradskii [51, pp. 122–123]
Cajori [13, vol. II, p. 101, §464]
Gibbs [30, Chapter XII]
Sklar [77, p. 130] on phase space
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