THE LOGARITHMIC INTEGRAL

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It is an important mathematical object in the theory of prime numbers and its use in number theory seems to first arise with Gauss. But it is also one of the first transcendental functions one runs into after the trigonometric and logarithmic functions: having classified the trigonometric and logarithmic functions as known, we then take integrals involving them and want to know whether those can be expressed as a "closed expression" involving just them. If we take the integral of $\log(t)$ from 1 to x we find that it is equal to $x\log(x)-x$, while if we take the integral of $1/\log(t)$ say from 0 to x we are not able to find any expression for it, and we may be led to call it $\mathrm{li}(x)$.

There is no paper in the literature that gives the history of the introduction of the logarithmic integral to analysis. Indeed it's well known that Gauss conjectured the prime number theorem which is stated in terms of the logarithmic integral, but what were the first publications in which the logarithmic integral appeared? What it a known object of analysis when Gauss made his conjecture? When I was reading on the history of the prime number theorem this is a question to which I couldn't find a single paper that gave a reliable answer.

The logarithmic integral is defined as

$$\mathrm{li}(x) = \lim_{\epsilon\to 0}\Big(\int_0^{1-\epsilon}\frac{dt}{\log t}+\int_{1+\epsilon}^x\frac{dt}{\log t}\Big).$$

The exponential integral is defined as

$$\mathrm{Ei}(x) = \lim_{\epsilon\to 0}\Big(\int_{-\infty}^{-\epsilon}\frac{e^{-t}}{t}dt+\int_{\epsilon}^x\frac{e^{-t}}{t}dt\Big).$$

1. Why is $\sin x$ an elementary function and $\mathrm{li} x$ isn't?

N. N. Lebedev, *Special functions and their applications*, 1972.

Rüdiger Thiele, *What is a function?*

D. T. Whiteside, *Patterns of Mathematical Thought in the later Seventeenth Century*

Computation of values of functions in tables. History of mathematical tables.

Cajori on notations for functions.

Companion encyclopedia of the history and philosophy of the Mathematical Sciences, volume 1, sect. 4.4.

Encyclopaedia Britannica, Thomas Spencer Baynes, p. 39, "function sui generis". I don't remember which edition.

Cayley's review of J. W. L. Glaisher's *Tables of the Numerical Values of the Sine-integral, Cosine-integral, and Exponential Integral*, p. 262 in the Proceedings

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of the Royal Society of London, From June 17, 1869 to June 16, 1870, vol. XVIII, 1870.

Are some functions more transcendental than others? For example, is some unclassified power series more transcendental than the power series for $\sin(x)$? What about Bessel functions?

2. Euler and his contemporaries

E421, E464, E475, E500, E521, E583, E620, E621, E629, E662.

Institutiones calculi integralis

Pietro Ferroni, *Magnitudinum exponentialium Logarithmorum*, 1782.

P. Mako S. J., 1768, Calculi differentialis et integralis institutio, p. 149

Lorenzo Mascheroni, *Adnotationes ad calculum integralem Euleri*, 1790, pp. 42ff.

P. Mako, S.J., *Calculi differentialis et integralis institutio*, 1768, p. 149.

Silvestre François Lacroix, *An elementary treatise on the differential and integral calculus*, (translated from the French), 1816, p. 239.

Other books that may discuss the logarithmic integral: Disqvisitiones analyticae maxime ad calcvlvm integralem et doctrinam ... By Johann Friedrich Pfaff, Principiorum calculi differentialis et integralis expositio elementaris By Simon Antoine J. L'Huilier,

3. Gauss and the prime number theorem

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See Ingham, *The distribution of prime numbers*

Charles James Hargreave, *Analytical Researches concerning Numbers*, The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science, Third Series, Vol. 35, No. 233, July 1849, p. 45.

Factor table for the sixth million: containing the least factor of every ... By James Glaisher, Introduction, p. 93

Augustus De Morgan, Library of Useful Knowledge, *The Differential and Integral Calculus*, Society for the Diffusion of Useful Knowledge, Baldwin and Cradock, London, 1842, p. 660.

Edmund Landau, *Der Integgrallogarithmus und die Zahlentheorie*, Rend. Circ. Matem. Palermo, t. XXIII, 1907, p. 126

4. Later authors in the mid 19th century century

Charles Brooke, *A synopsis of the principal formulae and results of pure mathematics*, 1829, p. 224. Cites LCD 427-37, Tr. L. 181-91, Hrsch, Int. Tab. LCD=Lacroix, Trait du Calcul Differentiel, Tr. L.=Translation of Lacroix, Hirsch=Meyer Hirsch Integral tables.

T. G. Hall, *Treatise on the Differential and Integral Calculus*, 1837, p. 338.

Auszüge aus einigen Briefen an der Professor Gilbert, aus mehreren Schreiben des Prof. Soldner zu München, Annalen der Physik, Neue Folge, Neunter Band, 1811, (old series Neun und Dreissigster Band), p. 239.

Johann Georg von Soldner, *Theorie et tables d'une nouvelle fonction transcendante*, 1809, Lindauer, München, p. 6.

Andreas von Ettingshausen, *Vorlesungen über die höhere Mathematik*, Erster Band, Carl Gerold, Wien, 1827, p. 365.

J. J. Littrow, *Anleitung zur höheren Mathematik*, Carl Gerold, Wien, 1836, p. 301.

R. Beez, *Beiträge zur Theorie des Integrallogarithmus*, pp. 419–441, Archiv der Mathematik und Physik, Neunzehnter Theil, 1852.

Rudolf Engelmann (ed.), *Abhandlungen von Friedrich Wilhelm Bessel*, Zweiter Band, Wilhelm Engelmann, Leipzig, 1876. Several contributions, starting p. 326.

Johann August Grunert, *Mathematisches Wörterbuch oder Erklärung der Begriffe, Lehrstätze, Aufgaben und Methoden der Mathematik*, Erste Abtheilung, Fünfter Theil, Erster Band, E. B. Schwickert, Leipzig, 1831, p. 138.

Johann August Grunert, *Elemente der Differential- und Integralrechnung zum Gebrauche bei Vorlesungen*, Zweiter Theil, E. B. Schwickert, Leipzig, 1837, p. 126

Oskar Schlömilch, *Beiträge zur Theorie bestimmter Integrale*, Friedrich Frommann, Jena, 1843, p. 70.

Oskar Schlömilch, *Zur Theorie des Integrallogarithmus*, Archiv der Mathematik und Physik, Neunter Theil, 1847, p. 5 and 307.

Hardy, *Divergent series*, p. 40.

Ferdinand Minding (ed.), *Handbuch der differential- und Integralrechnung und ihrer Anwendungen auf Geometrie zunächst zum Gebrauche in Vorlesungen*, F. Dümmler, Berlin, 1836, p. 100.

Car. Ant. Bretschneider, *Theoriae logarithmi integralis lineamenta nova*, p. 257, Journal für die reine und angewandte Mathematik, Siebenzehnter Band, 1837.

5. Liouville's theorem on integration in terms of elementary functions

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6. Later history

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