THE LOGARITHMIC INTEGRAL
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It is an important mathematical object in the theory of prime numbers and its use in number theory seems to first arise with Gauss. But it is also one of the first transcendental functions one runs into after the trigonometric and logarithmic functions: having classified the trigonometric and logarithmic functions as known, we then take integrals involving them and want to know whether those can be expressed as a “closed expression” involving just them. If we take the integral of $\log(t)$ from 1 to $x$ we find that it is equal to $x \log(x) - x$, while if we take the integral of $1/\log(t)$ say from 0 to $x$ we are not able to find any expression for it, and we may be led to call it $\text{li}(x)$.

There is no paper in the literature that gives the history of the introduction of the logarithmic integral to analysis. Indeed it’s well known that Gauss conjectured the prime number theorem which is stated in terms of the logarithmic integral, but what were the first publications in which the logarithmic integral appeared? What it a known object of analysis when Gauss made his conjecture? When I was reading on the history of the prime number theorem this is a question to which I couldn’t find a single paper that gave a reliable answer.

The logarithmic integral is defined as

$$\text{li}(x) = \lim_{\epsilon \to 0} \left( \int_{1-\epsilon}^{1+\epsilon} \frac{dt}{\log t} + \int_{1+\epsilon}^{x} \frac{dt}{\log t} \right).$$

The exponential integral is defined as

$$\text{Ei}(x) = \lim_{\epsilon \to 0} \left( \int_{-\infty}^{-\epsilon} \frac{e^{-t}}{t} dt + \int_{\epsilon}^{x} \frac{e^{-t}}{t} dt \right).$$

1. Why is $\sin x$ an elementary function and $\text{li}x$ isn’t?

N. N. Lebedev, Special functions and their applications, 1972.
Rüdiger Thiele, What is a function?
D. T. Whiteside, Patterns of Mathematical Thought in the later Seventeenth Century

Computation of values of functions in tables. History of mathematical tables.
Cajori on notations for functions.

Companion encyclopedia of the history and philosophy of the Mathematical Sciences, volume 1, sect. 4.4.

Encyclopaedia Britannica, Thomas Spencer Baynes, p. 39, “function sui generis”.
I don’t remember which edition.

Cayley’s review of J. W. L. Glaisher’s Tables of the Numerical Values of the Sine-integral, Cosine-integral, and Exponential Integral, p. 262 in the Proceedings

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Are some functions more transcendental than others? For example, is some unclassified power series more transcendental than the power series for \( \sin(x) \)? What about Bessel functions?

2. Euler and his contemporaries


Institutiones calculi integralis
P. Mako S. J., 1768, Calculi differentialis et integralis institutio, p. 149
Silvestre François Lacroix, *An elementary treatise on the differential and integral calculus*, (translated from the French), 1816, p. 239.

Other books that may discuss the logarithmic integral: *Disquisitiones analyticæ maxime ad calculum integrale et doctrinam ...* By Johann Friedrich Pfaff, Principleorum calculi differentialis et integralis expositio elementaris By Simon Antoine J. L’Huilier,

3. Gauss and the prime number theorem

Gauss *Werke*, Band 8, pp. 90–92.
See Ingham, *The distribution of prime numbers*


Factor table for the sixth million: containing the least factor of every ... By James Glaisher, Introduction, p. 93

4. Later authors in the mid 19th century century


*Auszüge aus einigen Briefen an der Professor Gilbert, aus mehreren Schreiben des Prof. Soldner zu München*, Annalen der Physik, Neue Folge, Neunter Band, 1811, (old series Neun und Dreissigster Band), p. 239.
5. LIOUVILLE’S THEOREM ON INTEGRATION IN TERMS OF ELEMENTARY FUNCTIONS

Liouville’s theorem in differential algebra.
Manuel Bronstein, Symbolic integration I: transcendental functions.
Brian Conrad, Impossibility theorems for elementary integration

6. LATER HISTORY

Detlef Laugwitz, Bernhard Riemann 1826-1866: turning points in the conception of mathematics.
Friedrich L. Bauer, Why Legendre made a wrong guess about \( \pi(x) \), and how Laguerre’s continued fraction for the logarithmic integral improved it, Math. Intelligencer, volume 25, number 3, 2003, pp. 7-11.
Julian Havil, Gamma: exploring Euler’s constant, p. 106.
Whittaker and Watson, A course of modern analysis, p. 341.