Homogeneous first order equations

Homogeneous first order equations: equations that can be written in terms of $x$ and $v$ when we do the substitution

$$v(x) = \frac{y(x)}{x}.$$ 

For example, $y' = x + y$ is not homogeneous but $y' = \frac{x}{y} + \frac{y}{x}$ is. The use of doing this substitution is that the equation becomes separable and we can then apply that method to solve the new equation for $v$ and then $y = xv$ solves the original equation.

e.g.

$$y' = \frac{x^2 + y^2}{xy}, \quad y(x_0) = y_0.$$ 

$v = \frac{y}{x}$. Then $y = xv$. Then using the product rule,

$$y' = v + xv'.$$

So the equation is

$$v + xv' = \frac{x^2 + y^2}{xy} = \frac{x^2}{xy} + \frac{y^2}{xy} = \frac{x}{y} + \frac{y}{x} = \frac{1}{v} + v,$$

$$v + xv' = \frac{1}{v} + v.$$ 

Then

$$xv' = \frac{1}{v}.$$ 

This is a separable equation.

$$vv' = \frac{1}{x}.$$ 

$$M(x) + N(v)v' = 0, \quad M(x) = -\frac{1}{x}, \quad N(v) = v.$$
\[ H'_1(x) = M(x) \text{ so } H_1(x) = -\ln |x|. \text{ And } H'_2(v) = N(v) \text{ so } H_2(v) = \frac{v^2}{2}. \] Then \( v \) satisfies

\[ H_1(x) + H_2(v(x)) = C. \]
\[ -\ln |x| + \frac{v(x)^2}{2} = C. \]
\[ v(x)^2 = 2 \ln |x| + 2C. \]

Generally, when an arbitrary constant \( C \) occurs it is valid to replace \( 2C \) with \( C \) as long as we replace every instance of \( 2C \) with \( C \). That is, if we had \( y + \frac{x}{C^2} = 2C \) i.e. \( y + \frac{2x}{C} = C \), but it is not equivalent with \( y + \frac{x}{C} = 2(C/2) \) i.e. \( y + \frac{x}{C} = C \). So

\[ v(x)^2 = 2 \ln |x| + C. \]

Our goal is to determine \( y \). Using \( v = \frac{y}{x} \),

\[ \frac{y^2}{x^2} = 2 \ln |x| + C. \]
\[ y(x)^2 = x^2(2 \ln |x| + C). \tag{1} \]

This is possible if and only if \( 2 \ln |x| + C \geq 0 \). In other words, to take the square root of the both sides of the above equation they must be nonnegative. Supposing \( 2 \ln |x| + C \geq 0 \) we take square roots:

\[ |y(x)| = |x| \sqrt{2 \ln |x| + C}. \tag{2} \]

It is not obvious at this point how to get an explicit formula for \( y(x) \). Each of

\[ y(x) = x \sqrt{2 \ln |x| + C}, \quad y(x) = -x \sqrt{2 \ln |x| + C}, \]
\[ y(x) = |x| \sqrt{2 \ln |x| + C}, \quad y(x) = -|x| \sqrt{2 \ln |x| + C} \]

satisfies the equation \(|y(x)| = |x| \sqrt{2 \ln |x| + C}\).

To narrow down the possible formulas for \( y \), use the fact that \( y \) is differentiable. This is a subtle point. The function \(|x|\) is not differentiable at \( x = 0 \): for \( x < 0 \), its derivative is \(-1\) and for \( x > 0 \) its derivative is \(1\), and at \( x = 0 \) its derivative does not exist. And whenever we talk about solutions of differential equations, the solution is differentiable, because the differential equation involves taking derivatives. So using the fact that \( y \) is differentiable, the only two possibilities for (2) are

\[ y(x) = x \sqrt{2 \ln |x| + C}, \quad y(x) = -x \sqrt{2 \ln |x| + C}. \]

For \( y(x_0) = y_0 \), using (1),

\[ y_0^2 = x_0^2(2 \ln |x_0| + C). \]
Then assuming \( x_0 \neq 0 \),
\[
\frac{y_0^2}{x_0} = 2 \ln |x_0| + C
\]
Then
\[
C = -2 \ln |x_0| + \frac{y_0^2}{x_0}.
\]
Therefore
\[
y(x) = x \sqrt{2 \ln |x| - 2 \ln |x_0| + \frac{y_0^2}{x_0}}, \quad y(x) = -x \sqrt{2 \ln |x| - 2 \ln |x_0| + \frac{y_0^2}{x_0}}. \tag{3}
\]
Using these formulas for \( y(x) \),
\[
y(x_0) = y_0, \quad y(x_0) = -y_0.
\]
Either \( y_0 = 0 \) or \( y_0 \neq 0 \). If \( y_0 = 0 \), then both of (3) are solutions of the initial value problem, because when \( y_0 = 0 \) then \( y_0 = -y_0 \) so both of \( y(x_0) = y_0 \) and \( y(x_0) = -y_0 \) are true.

If \( y_0 \neq 0 \), then only the first of (3) is a solution of the initial value problem, because for it \( y(x_0) = y_0 \) while for the other \( y(x_0) = -y_0 \).

Summarizing: if \( y_0 \neq 0 \) then there are two solutions of the initial value problem and if \( y_0 \neq 0 \) then there is a single solution of the initial value problem.

When solving the problem, I assumed that \( 2 \ln |x| + C \geq 0 \); this was to make sure that the argument of the square root is positive. I also assumed that \( x_0 \neq 0 \). We calculated \( C = -2 \ln |x_0| + \frac{y_0^2}{x_0} \), so \( 2 \ln |x| + C \geq 0 \) is equivalent with
\[
\ln |x| \geq -\frac{C}{2} = \ln |x_0| - \frac{1}{2} \frac{y_0^2}{x_0^2}. \]
That is,
\[
|x| \geq \exp \left( \ln |x_0| - \frac{1}{2} \frac{y_0^2}{x_0^2} \right).
\]
(If \( a > b \) then \( e^a > e^b \).) That is,
\[
|x| \geq |x_0| \exp \left( -\frac{1}{2} \frac{y_0^2}{x_0^2} \right).
\]
This is
\[
x \in \left( -\infty, -|x_0| \exp \left( -\frac{1}{2} \frac{y_0^2}{x_0^2} \right) \right] \cup \left[ |x_0| \exp \left( -\frac{1}{2} \frac{y_0^2}{x_0^2} \right), \infty \right)
\]
Only one of these intervals will be the domain of the solution: if \( x_0 \) is negative, then the first interval will be the domain, and if \( x_0 \) is positive, then the second interval will be the domain.
In class I did not write this out generally. Instead I did it with $x_0 = 1$ and $y_0 = 1$. Since $y_0 \neq 0$, here there is a single solution instead of two solutions: the solution is

$$y(x) = x\sqrt{2\ln|x| - 2\ln|x_0| + \frac{y_0^2}{x_0^2}} = x\sqrt{2\ln|x| + 1} = x\sqrt{2\ln x + 1}$$

and since $x_0 = 1$ is positive the domain of the solution is

$$|x_0|\exp\left(-\frac{1}{2}\frac{y_0^2}{x_0^2}\right), \infty) = [e^{-1/2}, \infty).$$

2 Exact equations

Exact equations generalize separable equations.

$$M(x, y) + N(x, y)y' = 0.$$ If $M_y = N_x$ then the equation is exact, and there is a function $F$ of two variables such that any solution $y(x)$ of the differential equation satisfies

$$F(x, y(x)) = C.$$ The function $F$ is defined by

$$F_x = M, \quad F_y = N.$$

3 Detailed example of determining domain of exact equation

I determine the domain in full detail. Every step is written out which makes it long.

$$1 - \frac{1}{yx^2} + \left(1 - \frac{1}{xy^2}\right)y' = 0, \quad y(x_0) = y_0.$$ Here $M = 1 - \frac{1}{yx^2}$ and $N = 1 - \frac{1}{xy^2}$. Then

$$M_y = \frac{1}{y^2x^2}, \quad N_x = \frac{1}{x^2y^2}.$$ So $M_y = N_x$ hence equation is exact.

To find $F$, as $F_x = M$,

$$F = \int M\,dx = \int 1 - \frac{1}{yx^2}\,dx = x + \frac{1}{yx} + g(y).$$
Using this,

\[ F_y = -\frac{1}{y^2x} + g'(y). \]

But

\[ F_y = N = 1 - \frac{1}{xy^2}, \]

so

\[-\frac{1}{y^2} + g'(y) = 1 - \frac{1}{xy^2}.\]

Hence \( g'(y) = 1 \), so

\[ g(y) = y. \]

Then

\[ F(x, y) = x + \frac{1}{yx} + g(y) = x + \frac{1}{yx} + y. \]

Therefore any solution \( y(x) \) of the differential equation satisfies

\[ x + \frac{1}{y(x)x} + y(x) = C \quad \text{(4)} \]

for some constant \( C \).

From (4) it follows that \( x = 0 \) does not belong to the domain of a solution of the differential equation, and also that if \( y \) is a solution of the differential equation, then for all \( x \), \( y(x) \neq 0 \).

Multiplying (4) by \( y \),

\[ xy + \frac{1}{x} + y^2 = Cy. \]

That is,

\[ y^2 + (x - C)y + \frac{1}{x} = 0. \]

So using the quadratic formula,

\[ y = \frac{-x + C \pm \sqrt{(x - C)^2 - \frac{4}{x}}}{2}. \]

Let

\[ y_1 = \frac{-x + C + \sqrt{(x - C)^2 - \frac{4}{x}}}{2}, \quad y_2 = \frac{-x + C - \sqrt{(x - C)^2 - \frac{4}{x}}}{2}. \]

The argument so far has established: if \( y \) is a solution of the initial value problem then either \( y = y_1 \) or \( y = y_2 \). We have not determined if exactly one of \( y_1, y_2 \) is a solution of the IVP or if both are.

From (4),

\[ x_0 + \frac{1}{y_0x_0} + y_0 = C. \]
Check if \( y_1, y_2 \) satisfy IVP. The following calculation applies to both \( y_1 \) and \( y_2 \):

\[
(x_0 - C)^2 - \frac{4}{x_0} = \left( \frac{-1}{y_0x_0} - y_0 \right)^2 - \frac{4}{x_0} = \frac{x_0^2}{y_0^2} + \frac{2y_0}{y_0x_0} + y_0^2 - \frac{4}{x_0} \]

\[
= \frac{1}{y_0^2} + \frac{2}{x_0} + y_0^2 - \frac{4}{x_0} \]

\[
= \left( \frac{1}{y_0x_0} - y_0 \right)^2.
\]

So

\[
\sqrt{(x_0 - C)^2 - \frac{4}{x_0}} = \sqrt{\left( \frac{1}{y_0x_0} - y_0 \right)^2} = \left| \frac{1}{y_0x_0} - y_0 \right|.
\]

Also,

\[-x_0 + C = -x_0 + x_0 + \frac{1}{y_0x_0} + y_0 = \frac{1}{y_0x_0} + y_0.
\]

Hence

\[y_1(x_0) = \frac{1}{y_0x_0} + y_0 + \frac{1}{y_0x_0} - y_0\]

\[y_2(x_0) = \frac{1}{y_0x_0} + y_0 - \frac{1}{y_0x_0} - y_0\]

Either \( \frac{1}{y_0x_0} - y_0 \) is negative, 0, or positive.

If \( \frac{1}{y_0x_0} - y_0 \) is negative: (as \( |x| = -x \) when \( x \) is negative, \( |x| = 0 \) when \( x = 0 \), and \( |x| = x \) when \( x \) is positive)

\[y_1(x_0) = \frac{1}{y_0x_0} + y_0 - \frac{1}{y_0x_0} + y_0 = y_0\]

\[y_2(x_0) = \frac{1}{y_0x_0} + y_0 + \frac{1}{y_0x_0} - y_0 = \frac{1}{y_0x_0}.
\]

Therefore: \( y_1(x_0) = y_0 \). We do not have enough information to determine if \( y_2(x_0) = y_0 \). It is the case that \( y_2 \) satisfies the IVP if \( \frac{1}{y_0x_0} = y_0 \) and does not satisfy the IVP if \( \frac{1}{y_0x_0} \neq y_0 \). For example, if \( x_0 = 1 \) and \( y_0 = 1 \) then \( \frac{1}{y_0x_0} = y_0 \), but if \( x_0 = 2 \) and \( y_0 = 2 \) then \( \frac{1}{y_0x_0} = \frac{1}{4} \neq y_0 \).

If \( \frac{1}{y_0x_0} - y_0 \) is 0: then because \( \frac{1}{y_0x_0} = y_0 \),

\[y_1(x_0) = \frac{1}{y_0x_0} + y_0\]

\[y_2(x_0) = \frac{1}{y_0x_0} + y_0\]

\[y_1(x_0) = \frac{y_0 + y_0}{2} = y_0\]
and

\[ y_2(x_0) = \frac{1}{\frac{1}{y_0} - x_0} + y_0 = \frac{y_0 + y_0}{2} = y_0. \]

Therefore: both \( y_1 \) and \( y_2 \) satisfy the IVP.

If \( \frac{1}{\frac{1}{y_0} - x_0} = y_0 \) is positive:

\[ y_1(x_0) = \frac{1}{\frac{1}{y_0} - x_0} + y_0 + \frac{1}{\frac{1}{y_0} - x_0} - y_0 = \frac{1}{y_0x_0} \]
\[ y_2(x_0) = \frac{1}{\frac{1}{y_0} - x_0} + y_0 - \frac{1}{\frac{1}{y_0} - x_0} + y_0 = y_0. \]

(This reverses the negative case.) Therefore: \( y_2 \) satisfies the IVP. If \( \frac{1}{\frac{1}{y_0} - x_0} = y_0 \) then \( y_1 \) satisfies the IVP, and if \( \frac{1}{\frac{1}{y_0} - x_0} \neq y_0 \) then \( y_1 \) does not satisfy the IVP.

We now determine the domains of solutions of the IVP. Let \( A(x) = (x - C)^2 - \frac{4}{x} \), so

\[ y_1(x) = \frac{-x + C + \sqrt{A(x)}}{2}, \quad y_2(x) = \frac{-x + C - \sqrt{A(x)}}{2}. \] (5)

The domain of \( y_1, y_2 \) is those \( x \) such that \( A(x) \) is either positive or 0. \( y_1 \) and \( y_2 \) have the same domain here.

To find where a function is positive and negative, find where it is 0 and the function will be positive and negative between the points where it is 0.

There is not a simple general way of stating what the roots of \( A(x) \) are. Given an initial value \( y(x_0) = y_0 \), this determines \( C \) and then the roots of \( A(x) \) can be calculated. Then given the roots of \( A(x) \), to find where \( A(x) \) is positive and negative, check the value of \( A(x) \) for \( x \) on either sides of the roots, and also check the value on either side of points where \( A(x) \) is undefined. For a continuous function to switch between positive and negative it must be 0 somewhere between, so between the points where it is 0 the function will be entirely positive or entirely negative.

**IVP** \( y(1) = 1 \). Using

\[ x + \frac{1}{y(x)x} + y(x) = C \]

and \( y(1) = 1 \),

\[ C = 3. \]

Then \( A(x) = (x - 3)^2 - \frac{4}{x} \).

\[ y_1(x) = \frac{-x + C + \sqrt{A(x)}}{2} = \frac{-x + 3 + \sqrt{A(x)}}{2}. \]
\[ y_2(x) = \frac{-x + C - \sqrt{A(x)}}{2} = \frac{-x + 3 - \sqrt{A(x)}}{2}. \]
Here \( \frac{1}{x_0 y_0} - y_0 = 0 \). By the earlier general calculations, both \( y_1 \) and \( y_2 \) satisfy the IVP. That is, we narrowed the possibilities for what the solutions are to the two functions \( y_1, y_2 \), and the initial value is satisfied by both of them. If the initial value was only satisfied by one, that would mean that it is the solution and the other function is not, but here because both functions satisfy \( y(1) = 1 \), they both solve the initial value problem.

Summarizing: the initial value problem

\[
1 - \frac{1}{y x^2} + \left( 1 - \frac{1}{x y^2} \right) y' = 0, \quad y(1) = 1
\]

has two solutions, using \( A(x) = (x - 3)^2 - \frac{4}{x} \):

\[
y_1(x) = \frac{-x + 3 + \sqrt{(x - 3)^2 - \frac{4}{x}}}{2} \quad y_2(x) = \frac{-x + 3 - \sqrt{(x - 3)^2 - \frac{4}{x}}}{2}.
\]

These are two different functions but they both pass through the point \((x_0, y_0) = (1, 1)\).

**Inequalities and the domain of a solution.** We worked out in the general argument above that the domain of \( y_1, y_2 \) is those \( x \) such that \( A(x) \geq 0 \), i.e. those \( x \) such that \( A(x) > 0 \) or \( A(x) = 0 \). To determine this, determine the \( x \) for which \( A(x) = 0 \). Then check at any points between where \( A(x) \) is 0 or undefined to determine if \( A(x) \) is positive or negative.

\( A(x) = 0 \) means \((x-C)^2 - \frac{4}{x} = 0\). This is equivalent with \( x^2 - 2Cx + C^2 - \frac{4}{x} = 0 \). This is equivalent \( x^3 - 2Cx^2 + C^2 x - 4 = 0 \).

Now for the IVP \( y(1) = 1 \), \( C = 3 \), so summarizing: \( x \) is in the domain of \( y_1, y_2 \) if and only if \( x^3 - 6x^2 + 9x - 4 = 0 \).

Let \( P(x) = x^3 - 6x^2 + 9x - 4 \).

Generally to find where a function is positive we need to first find where it is 0, that is find its roots. Now, \( P(x) \) is a cubic polynomial so to find its roots we have to guess one root, and if we can’t guess one then there is not much we can do. Usually they end up being some small \( x \), like \( x = -3 \) or \( x = 1 \). Here, if \( x = 1 \) then \( P(1) = 1 - 6 + 9 - 4 = 0 \). So \( x = 1 \) is a root of the polynomial \( P(x) \). To find the other roots of the polynomial we then divide the cubic by the factor \( x - 1 \). (If \( x = a \) is a root, then \( x - a \) is a factor.)

Polynomial long division:

\[
x - 1 \quad \overline{x^3 - 6x^2 + 9x - 4} \quad \frac{x^2 - 5x + 4}{-x^3 + x^2} \quad \overline{-5x^2 + 9x} \quad \frac{4x - 4}{5x^2 - 5x} \quad \overline{-4x + 4} \quad \frac{0}{0}
\]
That is, 
\[ x^3 - 6x^2 + 9x - 4 = (x - 1)(x^2 - 5x + 4). \]

And
\[ x^2 - 5x + 4 = (x - 1)(x - 4). \]

Therefore
\[ P(x) = (x - 1)(x - 1)(x - 4). \]

Therefore, \( A(x) = 0 \) for \( x = 1 \) and \( x = 4 \). And \( A(x) \) is undefined for \( x = 0 \).

So to determine where \( A(x) \) is positive and negative, we evaluate \( A(x) \) at points to the left and right of \( x = 0, 1, 4 \).

\( A(-1) = 20 \), positive. \( x = -1 \) was chosen because it is to the left of \( x = 0 \).

\( A(1/2) = -\frac{7}{4} \), negative. \( x = \frac{1}{2} \) was chosen because it is between \( x = 0 \) and \( x = 1 \). Any point between them could have been used.

\( A(2) = -1 \), negative.

\( A(5) = \frac{16}{5} \), positive.

So for \( x \in (-\infty, 0) \), \( A(x) > 0 \). For \( x \in (0, 1) \), \( A(x) < 0 \). For \( x \in (1, 4) \), \( A(x) < 0 \). For \( x \in (4, \infty) \), \( A(x) > 0 \).

Summarizing: \( A(x) > 0 \) for
\[ x \in (-\infty, 0) \cup (4, \infty). \]

\( A(x) < 0 \) for
\[ x \in (0, 1) \cup (1, 4). \]

\( A(x) = 0 \) for \( x = 0, 1, 4 \).

The domain of the solutions \( y_1, y_2 \) are those \( x \) such that \( A(x) \geq 0 \). Since the domain is demanded to be an interval, it will be one of
\[ (-\infty, 0), \quad [1, 1], \quad [4, \infty). \]

Here \( x_0 = 1 \), so the domain is the single point \( x = 1 \), written as \([1, 1]\) or \( \{1\}\). In other words, the integral curve through the point \((x_0, y_0) = (1, 1)\) does not pass through any other points.

## 4 Direction fields

**parametric curve** \( r(t) = (x(t), y(t)) \). In particular, \( r(t) = (t, y(t)) \) is the graph of the function \( y \). For example: (i) \( r(t) = (t, t^2) \) is another way of representing \( y = x^2 \), a parabola.

(ii) \( r(t) = (\cos t, \sin t) \) is another way of representing the circle of radius 1, \( x^2 + y^2 = 1 \). This uses the fact that \( \sin^2 t + \cos^2 t = 1 \) for all \( t \).

For a parametric curve \( r(t) \), the **tangent vector** is
\[ r'(t) = (x'(t), y'(t)). \]

In particular, when \( r(t) = (t, y(t)) \) then \( r'(t) = (1, y'(t)). \)
**Direction fields** Suppose \( y \) solves a differential equation \( y' = f(t, y) \). \( y \) is a function. Its graph is the parametric curve \( r(t) = (t, y(t)) \). The tangent vector is \( r'(t) = (1, y'(t)) \). And since we suppose \( y' = f(t, y) \), then \( r'(t) = (1, f(t, y(t))) \). This is stating the tangent vector to the curve \( (t, y(t)) \). The vector \( (1, f(t, y(t))) \) is called the **direction field** for the differential equation \( y' = f(t, y) \).

For a differential equation \( y' = f(t, y) \), when \( y \) solves this differential equation we call \( r(t) = (t, y(t)) \) an **integral curve** of the differential equation.

For example, \( y' + t^2y = \cos t \). Here \( y' = f(t, y) \) for \( f(t, y) = -t^2y + \cos t \). The vector \( (1, f(t, y(t))) = (1, -t^2y + \cos t) \) is the direction field for this differential equation.

In an initial value problem a differential equation \( y' = f(t, y) \) and an initial condition \( y(t_0) = t_0 \) are given. For \( y \) to be a solution of the differential equation is equivalent to \( (t, y(t)) \) being an integral curve of the differential equation. For \( y \) to satisfy the initial condition \( y'(t_0) = y_0 \) is equivalent to the point \( (t_0, y_0) \) lying on the curve \( (t, y(t)) \). Thus \( y \) solves the initial value problem if and only if \( (t, y(t)) \) is an integral curve going through the point \( (t_0, y_0) \).

### 5 Picard iterates

Initial value problem

\[
y'(t) = f(t, y(t)), \quad y(t_0) = y_0.
\]

Picard iterates

\[
\phi_0(t) = y_0
\]

\[
\phi_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, \phi_n(s))ds.
\]

Taking the derivative, using the fundamental theorem of calculus,

\[
\phi'_{n+1}(t) = f(t, \phi_n(t)).
\]

Supposing that the sequence of functions \( \phi_n \) converges to a function \( \phi \) (we have not precisely stated what it means to say that a sequence of functions converges to a function), then \( \phi'_{n+1} \to \phi' \) and \( \phi_n \to \phi \). Hence

\[
\phi'(t) = f(t, \phi(t)).
\]

That is, supposing that the sequence of Picard iterates converges to a function \( \phi \), the function \( y(t) = \phi(t) \) solves the initial value problem \( y'(t) = f(t, y(t)) \), \( y(t_0) = y_0 \).

**e.g.** \( y' + t^2y = \cos t \). For the moment we do not talk about integral curves, direction fields or Picard iterates. This example is a linear equation with \( p(t) = t^2 \) and \( g(t) = \cos t \). Then the integrating factor is

\[
\mu(t) = \exp(\int t^2 dt) = \exp(t^3/3).
\]
Then the **general solution** of the differential equation $y' + t^2 y = \cos t$ is

$$y(t) = \frac{1}{\mu(t)} \left( \int \mu(t) g(t) dt + C \right) = e^{-t^2/3} \left( \int e^{t^2/3} \cos t dt + C \right)$$

$$= e^{-t^2/3} \int e^{t^2/3} \cos t dt + Ce^{-t^2/3}.$$ 

There is no simple way of working out the integral.