Practice final exam
Term: Fall 2016

Student ID Information

Last name: ___________________________  First name: ___________________________

Student ID #: ________________________

<table>
<thead>
<tr>
<th>Course Code: MAT B44</th>
<th>Grade Table</th>
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<tbody>
<tr>
<td>Course Title: Differential Equations I</td>
<td>Question</td>
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<tr>
<td>Instructor: Jordan Bell</td>
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<tr>
<td>Date of Test: December 19</td>
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<td>Time Period: Start: 9:00 End: 11:50</td>
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<td>Duration of Test: 2 hours 50 minutes</td>
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<tr>
<td>Number of Test Pages: 17 pages</td>
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<td>(including this cover sheet)</td>
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<td>Additional Materials Allowed:</td>
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<tr>
<td>Scientific calculator</td>
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<td>Question</td>
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1. (6 points) Euler’s method

For the initial value problem
\[ y'(t) = \frac{y}{t} + \frac{1}{y}, \quad y(2) = 1, \]
use Euler’s method with step size \( h = 0.1 \) to approximate \( y(2.2) \).
2. (4 points) **Linear equations** Solve the initial value problem

\[ ty' + 2y = 4t^2, \quad y(1) = y_0. \]
3. (7 points) Separable equations

Find an explicit solution of the IVP

\[ y'(x) = \frac{3x^2}{3y-4}, \quad y(1) = 0. \]

Determine the domain of the solution.
4. (10 points) Homogeneous equations

Doing the substitution $v = \frac{y}{x}$, find the general solution of

$$x^2 y' = 3(x^2 + y^2) \arctan \frac{y}{x} + xy.$$
5. (7 points) Exact equations

Find an explicit solution of the IVP

\[ y + 2 + \left( -\frac{1}{y^2} + x \right) y' = 0, \quad y(1) = 2 \]

and determine the domain of the solution.
6. (5 points) Picard iterates

Calculate the Picard iterates \( \phi_0, \phi_1, \phi_2, \phi_3 \) for the IVP

\[
y' = t - y^2, \quad y(0) = 1.
\]
7. (4 points) **Wronskians** Let \( y_1, y_2 \) be two solutions of
\[
y'' + p(t)y' + q(t)y = 0,
\]
and let \( W = y_1 y_2' - y_1' y_2 \) be their Wronskian. Derive a differential equation that \( W \) satisfies.
8. (5 points) Reduction of order For the ODE
\[ y'' + py' + qy = 0, \]
let \( y_1 \) be a solution. It is a fact that \( y_2 = vy_1 \) is a solution of the ODE if and only if
\[ v' = \frac{1}{y_1^2} e^{-\int p \, dx}. \]

Given that \( y_1(x) = x \) is a solution of the ODE
\[ x^2 y'' + xy' - y = 0, \]
use the above formula for \( v' \) to calculate a solution \( y_2 = vy_1 \).
9. (6 points) Variation of parameters

Solve the initial value problem

\[ y'' - 2y' = 12t - 10, \quad y(0) = 0, \quad y'(0) = 1. \]

How does \( y \) behave as \( t \to \infty \)?
10. (6 points) Constant coefficient second order equations

Find the solution of the IVP

\[2y'' - 3y' + y = 0, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}\]

and find its maximum value.
11. (6 points) Complex eigenvalues

Find the solution of the initial value problem

\[ 2y'' + 2y' + 3y = 0, \quad y(0) = 0, \quad y'(0) = 1. \]
12. (6 points) Power series

Let \[ y = \sum_{n=0}^{\infty} a_n x^n \] be a solution of \[ y' + xy = 1. \]

Calculate \( a_0, a_1, a_2, a_3, a_4 \).
13. (6 points) Power series

Let $y = \sum_{n=0}^{\infty} a_n x^n$ be a solution of

$$(1 - x^2)y'' - xy' + \lambda^2 y = 0.$$ 

Calculate $a_0, a_1, a_2, a_3, a_4$. 
14. (6 points) Frobenius series

Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ be a solution of

$$5x^2 y'' + x(1 + x)y' - y = 0.$$ 

Find the indicial equation for $r$. For each of the roots of the indicial equation, calculate $a_0, a_1, a_2, a_3$. 
15. (6 points) Jordan canonical form

Use the matrix exponential to find the general solution of the ODE

\[ x' = \begin{pmatrix} \frac{3}{2} & \frac{3}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} x. \]
16. (11 points) Variation of parameters

Find the solution of the initial value problem

\[ x'(t) = \begin{pmatrix} 2 & -1 \\ -5 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 6e^t \\ -12e^t \end{pmatrix}, \quad x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \]