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Second order linear ODE

\[ y''(t) + p(t)y'(t) + q(t)y(t) = g(t). \]

It can be useful to write an ODE as \( L[y](t) = g(t) \) for

\[ L[y](t) = y''(t) + p(t)y'(t) + q(t)y(t), \]

in other words

\[ L[y] = y'' + py' + qy. \]

Example of second order ODE that is nonlinear:

\[ y'' + yy' + y = 0. \]

If \( p \) and \( q \) are constant, called constant coefficient and can be written as

\[ ay''(t) + by'(t) + cy(t) = g(t). \]

If \( g(t) = 0 \) then called homogeneous. (This is different than for first order ODE where homogeneous means doing the substitution \( v(t) = y(t)/t \).

The ODE being homogeneous is equivalent with \( y = 0 \) being a solution.

E.g.

\[ y''(t) + \sin(t)y'(t) + t^2y(t) = 0 \]

is not constant coefficient but is homogeneous.

We will work out a general method for solving constant coefficient homogeneous linear equations. We will do a few methods for solving nonlinear equations but there is not a general method that applies to any second order nonlinear ODE.

After we do homogeneous equations, we then do inhomogeneous. This uses the solution of homogeneous equations.

For a second order ODE, an IVP specifies \( y(t_0) = y_0, y'(t_0) = y_0' \). There will be two arbitrary constants in the general solution. There will be two linearly independent solutions of the ODE. Being linearly independent means that one is not a constant multiple of another: if \( y_1(t) \neq 0 \), then \( y_1(t), t \cdot y_1(t) \) are linearly independent but \( y_1(t), 2 \cdot y_1(t) \) are not linearly independent.
To solve
\[ ay'' + by' + cy = 0 \]
we want to find a general solution of the ODE that is real valued. The method to find the general solution first finds a general solution that is complex valued, and then turns this into a general solution that is real valued.

Let \( y(t) = e^{rt} \). It is not obvious that this is a solution of the equation. We find the values of \( r \) that make this a solution.

\[ y' = re^{rt}, \quad y'' = r^2 e^{rt}, \quad ay'' + by' + cy = ar^2 e^{rt} + br e^{rt} + ce^{rt} = e^{rt}(ar^2 + br + c). \]

\( y'(t) = 0 \) if and only if \( e^{rt}(ar^2 + br + c) = 0 \) if and only if \( ar^2 + br + c = 0. \)

\[ ar^2 + br + c = 0 \]
is called the characteristic equation of the ODE \( ay'' + by' + cy = 0. \)

\( r \) is a root of the characteristic equation if and only if \( r = r_1 \) or \( r = r_2, \)

\[ r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \]

Let \( D = b^2 - 4ac. \) Either \( D < 0, \) \( D = 0, \) \( D > 0. \)

If \( D < 0, \) then \( r_1, r_2 \) are complex because they each have \( \sqrt{D}. \) We will calculate \( y_1(t) = e^{r_1t}, \ y_2(t) = e^{r_2t}. \)

If \( D = 0, \) then \( r_1 = \frac{b}{2a}, r_2 = \frac{-b}{2a} \) so are equal. We will calculate \( y_1(t) = e^{r_1t} = e^{-\frac{b}{2a}t}. \) Then we calculate an independent solution \( y_2 \) using reduction of order, which we will explain shortly. This method of reduction of order can also be used to calculate \( y_2 \) for the cases \( D < 0 \) and \( D > 0 \) and it gives the same solutions that we find for them. But for those cases we do not have to use reduction of order, while for \( D = 0 \) we do have to use it.

If \( D > 0, \) then \( r_1, r_2 \) are real numbers and are not equal, since \( \sqrt{D} > 0 \) and one adds it while one subtracts it in the quadratic formula.

**Case** \( D < 0, \) \( D = b^2 - 4ac. \) For example,

\[ y'' - y' + y = 0. \]

\( a = 1, \ b = -1, \ c = 1. \) \( D = 1 - 4 = -3, \) so \( D < 0. \) The roots of the characteristic equation are

\[ r_1 = \frac{1 + \sqrt{-3}}{2} = \frac{1}{2} + \frac{i \sqrt{3}}{2}, \quad r_2 = \frac{1 - \sqrt{-3}}{2} = \frac{1}{2} - \frac{i \sqrt{3}}{2}. \]

We use the general formula

\[ e^{i\theta} = \cos \theta + i \sin \theta. \]

And we will use the facts \( \cos(-\theta) = \cos(\theta), \sin(-\theta) = -\sin(\theta). \)
Then using $e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y)$,

$$y_1(t) = e^{r_1 t} = e^z e^{i \frac{\sqrt{3}}{2} t} = e^z \left( \cos \left( \frac{\sqrt{3}}{2} t \right) + i \sin \left( \frac{\sqrt{3}}{2} t \right) \right).$$

$$y_2(t) = e^{r_2 t} = e^z e^{-i \frac{\sqrt{3}}{2} t} = e^z \left( \cos \left( -\frac{\sqrt{3}}{2} t \right) + i \sin \left( -\frac{\sqrt{3}}{2} t \right) \right) = e^z \left( \cos \left( \frac{\sqrt{3}}{2} t \right) - i \sin \left( \frac{\sqrt{3}}{2} t \right) \right).$$

To see if $y_1$ and $y_2$ are linearly independent, check if $\frac{y_1}{y_2}$ is constant.

$$\frac{y_1}{y_2} = \frac{e^{r_1 t}}{e^{r_2 t}} = e^{r_1 t - r_2 t} = e^{(r_1 - r_2) t}.$$ Generally,

$$r_1 - r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} - \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{\sqrt{b^2 - 4ac}}{a} = \frac{\sqrt{D}}{a}.$$ Here,

$$r_1 - r_2 = 2 \cdot i \frac{\sqrt{3}}{2} = i \sqrt{3}.$$ So

$$\frac{y_1}{y_2} = e^{i \frac{\sqrt{3}}{2}},$$

which is not constant. This depends on $r_1 - r_2 \neq 0$, because if $r_1 - r_2 = 0$ then $\frac{y_1}{y_2} = e^{0 \cdot t} = 1$.

Because in this case $D < 0$, in particular $D \neq 0$ hence $r_1 - r_2 \neq 0$. For $y'' - y' + y = 0$, $r_1 - r_2 = i \sqrt{3}$.

So $y_2$ is not a constant multiple of $y_1$. This is true generally for the case $D < 0$. Then a general solution with complex values for this case is

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{r_1 t} + c_2 e^{r_2 t}, \quad c_1, c_2 \in \mathbb{C}.$$ But because we want a general solution with real values, we cannot directly use $e^{r_1 t} = \cos(r_1 t) + i \sin(r_1 t)$ and $e^{r_2} = \cos(r_2 t) + i \sin(r_2 t)$ because each of these takes complex values.

Instead of using $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$, we work out that the real and imaginary parts of any solution are themselves solutions, and then take the real
and imaginary parts of $y_1$ as two independent solutions. We can also take the real and imaginary parts of $y_2$ instead of $y_1$, and this will give the same solutions as we get from $y_1$.

Generally, if $y = u + iv$, then, as $y' = u' + iv'$ and $y'' = u'' + iv''$, 

$$y'' + py + qy = 0$$

if and only if 

$$(u'' + pu' + qu) + i(v'' + pv' + qv) = 0.$$ 

But $\alpha + i\beta = 0$ if and only if $\alpha = 0$ and $\beta = 0$. So for $y = u + iv$, 

$$y'' + py' + qy = 0$$

if and only if 

$$u'' + pu' + qu = 0, \quad v'' + pv' + qv = 0.$$ 

So let $y_1 = u + iv$: Here, 

$$y_1 = e^{rt}$$

$$= e^{t(\frac{1}{2} + i\frac{\sqrt{3}}{2})}$$

$$= e^{t/2} e^{i\frac{\sqrt{3}}{2}t}$$

$$= e^{t/2} \left( \cos \left( \frac{\sqrt{3}}{2}t \right) + i \sin \left( \frac{\sqrt{3}}{2}t \right) \right)$$

$$= e^{t/2} \cos \left( \frac{\sqrt{3}}{2}t \right) + ie^{t/2} \sin \left( \frac{\sqrt{3}}{2}t \right).$$

That is 

$$u(t) = e^{t/2} \cos \left( \frac{\sqrt{3}}{2}t \right), \quad v(t) = e^{t/2} \sin \left( \frac{\sqrt{3}}{2}t \right).$$

Because $y_1$ satisfies the ODE $y'' - y' + y = 0$, $u$ and $v$ also both satisfy this ODE. $u$ and $v$ are real valued.

Generally, for $ay'' + by' + cy = 0$ and $D = b^2 - 4ac$, $r_1 = \frac{-b + \sqrt{D}}{2a}$ and $r_2 = \frac{-b - \sqrt{D}}{2a}$. In the case $D < 0$, which is the case we are working on, $r_1 = \frac{-b + \sqrt{-D}}{2a}$, $r_2 = \frac{-b - i\sqrt{-D}}{2a}$. $D < 0$ implies $-D > 0$. 

$$e^{rt} = e^{-\frac{b}{2a}t} e^{i\frac{\sqrt{-D}}{2a}t}$$

$$= e^{-\frac{b}{2a}t} \left( \cos \left( \frac{\sqrt{-D}}{2a}t \right) + i \sin \left( \frac{\sqrt{-D}}{2a}t \right) \right)$$

So 

$$u(t) = e^{-\frac{b}{2a}t} \cos \left( \frac{\sqrt{-D}}{2a}t \right), \quad v(t) = e^{-\frac{b}{2a}t} \sin \left( \frac{\sqrt{-D}}{2a}t \right).$$
Then
\[
\frac{u(t)}{v(t)} = \frac{e^{-\frac{t}{\sqrt{2a}}} \cos \left( \frac{\sqrt{-D}}{2a} t \right)}{e^{-\frac{t}{\sqrt{2a}}} \sin \left( \frac{\sqrt{-D}}{2a} t \right)} = \frac{\cos \left( \frac{\sqrt{-D}}{2a} t \right)}{\sin \left( \frac{\sqrt{-D}}{2a} t \right)} = \cot \left( \frac{\sqrt{-D}}{2a} t \right).
\]

Since \( D \neq 0 \) in this case, \( \frac{v}{u} \) is not constant so \( u \) and \( v \) are linearly independent solutions of \( ay'' + by' + cy = 0 \).

**Summary:** For the case \( D < 0 \), \( D = b^2 - 4ac \), the general solution of \( ay'' + by' + cy = 0 \) is
\[
y(t) = c_1 e^{-\frac{t}{\sqrt{2a}}} \cos \left( \frac{\sqrt{-D}}{2a} t \right) + c_2 e^{-\frac{t}{\sqrt{2a}}} \sin \left( \frac{\sqrt{-D}}{2a} t \right), \quad c_1, c_2 \in \mathbb{R}.
\]

For the case \( D < 0 \), we in fact only have to write out one of \( y_1, y_2 \). Usually I write out \( y_1 \) and take its real and imaginary parts, but doing this for \( y_2 \) will also work.

For IVP \( y(t_0) = y_0, \ y'(t_0) = y'_0 \), we express \( c_1, c_2 \) in terms of \( t_0, y_0, y'_0 \).

For the ODE \( y'' - y' + y = 0 \) these give the real linearly independent solutions we calculated earlier. The general solution is
\[
y(t) = c_1 e^{t/2} \cos \left( \frac{\sqrt{3}}{2} t \right) + c_2 e^{t/2} \sin \left( \frac{\sqrt{3}}{2} t \right), \quad c_1, c_2 \in \mathbb{R}.
\]

For IVP \( y(0) = y_0 \) and \( y'(0) = y'_0 \), \( y'' - y' + y = 0 \), calculate \( c_1, c_2 \).

\[
y'(t) = \frac{1}{2} c_1 e^{t/2} \cos \left( \frac{\sqrt{3}}{2} t \right) - \frac{\sqrt{3}}{2} c_1 e^{t/2} \sin \left( \frac{\sqrt{3}}{2} t \right)
+ \frac{1}{2} c_2 e^{t/2} \sin \left( \frac{\sqrt{3}}{2} t \right) + \frac{\sqrt{3}}{2} c_2 e^{t/2} \cos \left( \frac{\sqrt{3}}{2} t \right)
\]

As \( e^0 = 1, \cos(0) = 1 \) and \( \sin(0) = 0 \),
\[
y_0 = y(t_0) = y(0) = c_1.
\]
\[
y'_0 = y'(t_0) = y'(0) = \frac{1}{2} c_1 + \frac{\sqrt{3}}{2} c_2.
\]

Then
\[
y'_0 = \frac{1}{2} y_0 + \frac{\sqrt{3}}{2} c_2
\]
implies
\[
\frac{\sqrt{3}}{2} c_2 = y'_0 - \frac{1}{2} y_0
\]
implies
\[
\sqrt{3} c_2 = 2 y'_0 - y_0
\]
implies
\[ c_2 = \frac{2y_0' - y_0}{\sqrt{3}}. \]

So the solution of the IVP \( y'' - y' + y = 0, \ y(0) = y_0, \ y'(0) = y_0', \) is
\[ y(t) = y_0 e^{t/2} \cos \left( \frac{\sqrt{3}}{2} t \right) + \left( \frac{2y_0' - y_0}{\sqrt{3}} \right) e^{t/2} \sin \left( \frac{\sqrt{3}}{2} t \right). \]

**Case** \( D = 0, \ D = b^2 - 4ac. \) We do this case using reduction of order, and we present reduction of order after the case \( D > 0. \) One solution is \( y_1(t) = e^{b/2t}, \) and a linearly independent solution will be found of the form \( y_2(t) = v(t)y_1(t). \)

**Case** \( D > 0, \ D = b^2 - 4ac. \) e.g.
\[ y'' + y' - y = 0. \]

\( D = b^2 - 4ac = 1 + 4 = 5. \)

The characteristic equation is \( r^2 + r - 1 = 0. \) The roots are
\[ r_1 = \frac{-b + \sqrt{D}}{2a}, \quad r_2 = \frac{-b - \sqrt{D}}{2a}. \]

Because in this case \( D > 0, \) \( r_1 \) and \( r_2 \) are real and \( r_1 \neq r_2, \) namely they are distinct real roots. Let
\[ y_1(t) = e^{r_1t} = e^{\frac{-b + \sqrt{D}}{2a} t}, \quad y_2(t) = e^{r_2t} = e^{\frac{-b - \sqrt{D}}{2a} t}. \]

These are real valued solutions of \( ay'' + by' + cy = 0. \) For \( y_1, y_2 \) to be linearly independent, check if \( \frac{y_1}{y_2} \) is constant.
\[ \frac{y_1}{y_2} = \frac{e^{\frac{-b + \sqrt{D}}{2a} t}}{e^{\frac{-b - \sqrt{D}}{2a} t}} = e^{\frac{-b + \sqrt{D}}{2a} t - \frac{-b - \sqrt{D}}{2a} t} = e^{\frac{\sqrt{D}}{2a} t}. \]

Since \( D \neq 0, \) \( \frac{y_1}{y_2} \) is not constant. 

**Summary:** For \( D > 0, \) the general solution of \( ay'' + by' + cy = 0 \) is
\[ y(t) = c_1 e^{\frac{-b + \sqrt{D}}{2a} t} + c_2 e^{\frac{-b - \sqrt{D}}{2a} t}, \quad c_1, c_2 \in \mathbb{R}. \]

That is,
\[ y(t) = c_1 e^{r_1t} + c_2 e^{r_2t}, \quad c_1, c_2 \in \mathbb{R}. \]

For \( y'' + y' - y = 0, \) \( a = 1, b = 1, c = -1, D = 5, \) the general solution is
\[ y(t) = c_1 e^{\frac{-1 + \sqrt{5}}{2} t} + c_2 e^{\frac{-1 - \sqrt{5}}{2} t}. \]

For IVP \( y(0) = y_0, \ y'(0) = y_0', \ y'' + y' - y = 0, \) calculate \( c_1, c_2. \)
\[ y'(t) = c_1 r_1 e^{r_1t} + c_2 r_2 e^{r_2t}. \]
\( y_0 = y(0) = c_1 + c_2, \quad y_0' = y'(0) = c_1 r_1 + c_2 r_2. \)

\( c_1 = y_0 - c_2. \) Then
\[
y_0' = (y_0 - c_2) r_1 + c_2 r_2
\]
then
\[
y_0' = y_0 r_1 + c_2 (-r_1 + r_2)
\]
then
\[
c_2 (-r_1 + r_2) = y_0' - y_0 r_1
\]
then
\[
c_2 = \frac{y_0' - y_0 r_1}{-r_1 + r_2}.
\]
This expresses \( c_2 \) in terms of the initial values. Then
\[
c_1 = y_0 - c_2 = y_0 - \frac{y_0' - y_0 r_1}{-r_1 + r_2} = \frac{-y_0 r_1 + r_2 y_0 - y_0' + y_0 r_1}{-r_1 + r_2} = \frac{r_2 y_0 - y_0'}{-r_1 + r_2}.
\]
That is, multiplying numerator and denominator by \(-1\),
\[
c_1 = \frac{-r_2 y_0 + y_0'}{r_1 - r_2} = \frac{y_0' - r_2 y_0}{r_1 - r_2}.
\]
Therefore the solution of the IVP \( y(0) = y_0, y'(0) = y_0', y'' + y' - y = 0 \) is
\[
y(t) = \left( \frac{y_0' - r_2 y_0}{r_1 - r_2} \right) e^{-\frac{1 + \sqrt{5}}{2} t} + \left( \frac{y_0' - y_0 r_1}{-r_1 + r_2} \right) e^{-\frac{1 - \sqrt{5}}{2} t}.
\]
where
\[
r_1 = \frac{-1 + \sqrt{5}}{2}, \quad r_2 = \frac{-1 + \sqrt{5}}{2}.
\]

Reduction of order. Let
\[
L[y] = y'' + py' + qy.
\]
Say \( L[y_1] = 0 \). Let \( y_2 = vy_1 \).

\[
y_2'' = v'' y_1 + v y_1' + vy_1' + vy_1'' = v'' y_1 + 2v' y_1' + vy_1''.
\]
Then
\[
L[y_2] = y_2'' + py_2' + qy_2
= v'' y_1 + 2v' y_1' + vy_1'' + p(v' y_1 + vy_1') + q v y_1
= v(y'' + py_1' + qy_1) + v'' y_1 + 2v' y_1' + pv' y_1
= v L[y_1] + v' (2y_1' + py_1) + v'' y_1.
\]
Because we have assumed \( L[y_1] = 0 \),
\[
L[y_2] = v' (2y_1' + py_1) + v'' y_1.
\]
Therefore $L[y_2] = 0$ if and only if

$v'(2y_1' + py_1) + v''y_1 = 0$

if and only if

$v''y_1 = -v'(2y_1' + py_1)$

if and only if (assuming $v' \neq 0$ and $y_1 \neq 0$)

$v'' = \frac{-2y_1' - py_1}{y_1}$

if and only if, as $(\ln f)' = \frac{f'}{f}$,

$\frac{v''}{v'} = -\frac{2y_1'}{y_1} - p$

if and only if

$(\ln v')' = -2(\ln y_1)' - p$.

Integrating, as $\int (\ln f)' dt = \ln f$,

$\ln v' = -2 \ln y_1 - \int p(t) dt$.

That is,

$\ln v' = \ln \frac{1}{y_1^2} - \int p(t) dt$.

Therefore

$v' = e^{\ln \frac{1}{y_1^2} - \int p(t) dt} = \frac{1}{y_1^2} e^{-\int p(t) dt}$.

**Summary:** For $L[y] = y'' + py' + qy$, if $L[y_1] = 0$, let $y_2 = vy_1$. $L[y_2] = 0$ if and only if

$v'(t) = \frac{1}{y_1^2} e^{-\int p(t) dt}$.

We cannot simplify this more generally. That is, if $y_1$ is solution of $y'' + py' + qy = 0$ then $y_2 = vy_1$ is solution when $v$ is determined by above formula.

For $y_2 = vy_1$, $y_1$ and $y_2$ being linearly independent means that $v$ is not constant. But for $v$ to be constant means that $v' = 0$, and the above formula for $v'$ cannot be 0 because $\frac{1}{y_1}$ cannot be 0 and $e^a$ cannot be 0. So $y_2 = vy_1$ given by reduction of order is linearly independent.

**Example** $x^2 y'' + xy' - y = 0$, i.e. $y'' + \frac{1}{x} y' - \frac{1}{x^2} y = 0$. $L[y] = y'' + \frac{1}{x} y' - \frac{1}{x^2} y$, $p = \frac{1}{x}$, $q = -\frac{1}{x^2}$.

Given that $y_1(x) = x$ is a solution, that is given that $L[y_1] = 0$, use reduction of order to find another solution.

$v' = \frac{1}{x^2} e^{-\int \frac{1}{x} dx}$. 

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Assume that $x > 0$ so we can ignore absolute values. If $x < 0$ then sometimes we can still ignore absolute values because they often cancel out in pairs. (This is not a precise explanation. I will state in questions when I want you to do separate cases for $x < 0$ and $x > 0$ for absolute values.) Then

$$v' = \frac{1}{x^2} e^{-\ln x} = \frac{1}{x^2} e^{\ln \frac{1}{x}} = \frac{1}{x^2} \cdot \frac{1}{x} = \frac{1}{x^3} = x^{-3}.$$ 

Then

$$v = -\frac{1}{2x^2}.$$

So

$$y_1 = x$$

is given, and by reduction of order

$$y_2 = vy_1 = -\frac{1}{2x^2} \cdot x = -\frac{1}{2x}.$$

Therefore the general solution of $x^2 y'' + xy' - y = 0$ is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 x - \frac{c_2}{2x}, \quad c_1, c_2 \in \mathbb{R}.$$ 

Because $c_2$ is an arbitrary constant, replacing $-\frac{c_2}{2}$ with $c_2$, we can also write the general solution as

$$y(t) = c_1 x + \frac{c_2}{x}.$$ 

**Example** $t^2 y'' + 2ty' - 2y = 0$. To put in form $y'' + py' + qy = 0$, divide by $t^2$: $L[y] = y'' + \frac{2}{t^2} y' - \frac{2}{t^2} y$. Given that $y_1 = t$ is solution, i.e. $L[y_1] = 0$, find another solution $y_2$.

Using reduction of order $y_2 = vy_1$, $v$ is determined by

$$v'(t) = \frac{1}{y_1^2} e^{-\int p(t) dt} = \frac{1}{t^2} e^{-\int \frac{2}{t^2} dt} = \frac{1}{t^2} e^{-2\ln t} = \frac{1}{t^2} e^{\ln \frac{1}{t^2}} = \frac{1}{t^2} \cdot \frac{1}{t^2} = t^{-4}.$$ 

(Assuming we can use $\ln t$ instead of $\ln |t|$.) Then integrating

$$v(t) = \frac{t^{-3}}{-3} = -\frac{1}{3t^3}.$$ 

Then

$$y_2(t) = v(t)y_1(t) = -\frac{1}{3t^3} \cdot t = -\frac{1}{3t^2}.$$ 

Therefore, given that $y_1(t) = t$ is solution, another solution is $y_2(t) = -\frac{1}{3t^2}$.

Therefore the general solution of $t^2 y'' + 2ty' - 2y = 0$ is

$$y(t) = c_1 t - \frac{c_2}{3t^2}, \quad c_1, c_2 \in \mathbb{R},$$

equivalently,

$$y(t) = c_1 t + \frac{c_2}{t^2}, \quad c_1, c_2 \in \mathbb{R}.$$
**Wronskian.** $L[y] = y'' + py' + qy$. IVP $L[y] = 0$, $y(t_0) = y_0, y'(t_0) = y'_0$.

Say general solution is

$$y(t) = c_1y_1(t) + c_2y_2(t), \quad c_1, c_2 \in \mathbb{R}.$$ 

Then

$$y'(t) = c_1y'_1(t) + c_2y'_2(t).$$

and

$$y_0 = y(t_0) = c_1y_1(t_0) + c_2y_2(t_0)$$

and

$$y'_0 = y'(t_0) = c_1y'_1(t_0) + c_2y'_2(t_0)$$

We can write this uses matrices as

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}.$$ 

Let

$$A(t) = \begin{pmatrix} y_1(t) & y_2(t) \\ y'_1(t) & y'_2(t) \end{pmatrix}.$$ 

The above equation is

$$A(t_0) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix},$$ 

The Wronskian of the ODE $y'' + py' + qy = 0$ is

$$W(t) = \det \begin{pmatrix} y_1(t) & y_2(t) \\ y'_1(t) & y'_2(t) \end{pmatrix} = y_1(t)y'_2(t) - y_2(t)y'_1(t).$$

We can determine $c_1, c_2$ in terms of $y_0, y'_0$ if $W(t_0) \neq 0$, as this means that $A(t_0)$ is invertible. If $W(t_0) = 0$ then either there will be more than one choice of $c_1, c_2$ or there will be no choices of $c_1, c_2$.

It does not have to be clear yet why the Wronskian is worth talking about.

Later when we do systems of linear differential equations, we will talk about the Wronskian of a system of $n$ first order linear differential equations. The Wronskian we are talking about for second order differential equations will be equivalent to the Wronskian of $n = 2$ differential equations.

For systems of $n$ linear differential equations, we will use Cramer’s rule.

For $A$ an $n \times n$ matrix, $b \in \mathbb{R}^n$, $x \in \mathbb{R}^n$,

$$Ax = b.$$ 

Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$.

Let $A = [a_1 \cdots a_n]$. $a_j \in \mathbb{R}^n$ is the $j$th column of $A$. We can write

$$a_j = Ae_j.$$ 

Let

$$X_k = [e_1 \cdots x \cdots e_n]$$
where the vector $x \in \mathbb{R}^n$ is put in the $k$th column.

$$AX_k = [Ae_1 \cdots Ax \cdots Ae_n] = [a_1 \cdots Ax \cdots a_n] = M_k,$$

with the vector $b \in \mathbb{R}^n$ in the $k$th column.

$$\det(AX_k) = \det(M_k) \implies \det(A) \det(X_k) = \det(M_k) \implies$$

$$\det(X_k) = \frac{\det M_k}{\det A}.$$

It can be calculated that $\det X_k = (-1)^{k+1}x_k$ where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. So

$$x_k = (-1)^{k+1} \det X_k = (-1)^{k+1} \cdot \frac{\det M_k}{\det A}.$$

This gives a formula for the entries of $x$ satisfying $Ax = b$, in terms of the entries of the matrix $A$ and the entries of the vector $b$.

Now apply Cramer’s rule to

$$A(t_0) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

$$A(t_0) = \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix}.$$ 

Apply Cramer’s rule with $x = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ and $b = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$.

$M_1 = \begin{pmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{pmatrix}$ and $M_2 = \begin{pmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{pmatrix}$.

$$\det M_1 = y_0y_2'(t_0) - y_2(t_0)y_0', \quad \det M_2 = y_1(t_0)y_0' - y_0y_1'(t_0).$$

$$W(t_0) = \det A(t_0).$$

$$c_1 = (-1)^2 \cdot \frac{\det M_1}{\det A} = \frac{y_0y_2'(t_0) - y_2(t_0)y_0'}{W(t_0)}$$

and

$$c_2 = (-1)^3 \cdot \frac{\det M_2}{\det A} = \frac{y_1(t_0)y_0' - y_0y_1'(t_0)}{W(t_0)} = \frac{y_0y_1'(t_0) - y_1(t_0)y_0'}{W(t_0)}.$$ 

$$W(t_0) = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0).$$

**Summary** We stated Cramer’s rule. The Wronskian occurs in the denominator of the formula it gives us. This formula expresses $c_1, c_2$ in terms of $y_0, y_0'$. 

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We did not have to use Cramer’s rule here with second order differential equations. We could get the above formulas for \( c_1, c_2 \) in terms of \( y_0, y_0' \) by rearranging the two equations we get by applying the initial values. We would get the same formulas for \( c_1, c_2 \). But we will use the above for systems of \( n \) differential equations, where the general solution has arbitrary constants \( c_1, \ldots, c_n \) that are determined using \( n \) initial values.

**Reduction of order for** \( D = 0 \): For \( ay'' + by' + cy = 0 \) with \( D = b^2 - 4ac = 0 \), the roots of the characteristic equation \( ar^2 + br + c = 0 \) are \( r_1 = \frac{-b}{2a}, r_2 = \frac{-b}{2a} \), which are equal. Let

\[
L[y] = y'' + py' + qy, \quad p = \frac{b}{a}, q = \frac{c}{a}.
\]

\( y_1(t) = e^{rt} = e^{-\frac{bt}{2a}} \) satisfies \( L[y_1] = 0 \).

To find another solution use reduction of order. \( y_2 = vy_1 \). \( L[y_2] = 0 \) if and only if

\[
v'(t) = \frac{1}{y_1^2} e^{-\int p(t)dt}.
\]

We worked this out earlier. As \( p = \frac{b}{a} \) and \( y_1 = e^{-\frac{bt}{2a}} \),

\[
v'(t) = e^{\frac{bt}{a}} e^{-\int \frac{b}{a} dt} = e^{\frac{bt}{a}} e^{-\frac{bt}{2a}} = e^{0} = 1.
\]

Then integrating

\[v(t) = t.
\]

Because we are just trying to find a single nonzero function \( v \), we do not need to add an arbitrary constant when integrating \( v' \).

**Summary:** For \( ay'' + by' + cy = 0 \), \( D = b^2 - 4ac = 0 \), one solution is \( y_1(t) = e^{-\frac{bt}{2a}} \). Using reduction of order, another solution is \( y_2(t) = vy_1 = te^{-\frac{bt}{2a}} \). These solutions are linearly independent. In other words, when the characteristic equation \( ar^2 + br + c = 0 \) has repeated roots \( r_1 = r_2 \), one solution is \( y_1(t) = e^{rt} \) and another solution is \( y_2(t) = te^{rt} \).

Remark: it is not true for the cases \( D < 0 \) or \( D > 0 \) that \( ty_1 \) is also a solution if \( y_1 \) is a solution. We got \( v = t \) using the particular facts about the case \( D = 0 \) which do not apply to \( D < 0 \) or \( D > 0 \).

**Solving linear or nonlinear equations by doing substitution to reduce to a first order differential equation** If \( y \) does not occur in a second order differential equation but only \( y' \) and \( y'' \), then take \( z = y' \). Then \( z' = y'' \) and we have first order differential equation for \( z \).

**Example** \( xy'' - y' = 3x^2 \). This is linear, not constant coefficient, and inhomogeneous. \( z = y' \) it becomes

\[xz' - z = 3x^2.
\]

This is a first order linear ODE: \( z' - \frac{1}{x}z = 3x, p = -\frac{1}{x} \) and \( g = 3x \). Here \( p \) and \( g \) refer to the first order ODE not the second order ODE.

\[
\mu(x) = e^\int p(x)dx = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = x^{-1}.
\]
Solution of first order ODE is, using method for linear first order ODE,

\[ z(x) = \frac{1}{\mu(x)} \left( \int \mu(x) g(x) dx + C_1 \right) = x \left( \int \frac{3x}{x} dx + C_1 \right) = x(3x+C_1) = 3x^2+C_1x. \]

Then we calculate \( y \) from \( z = y' \): the general solution of \( xy'' - y' = 3x^2 \) is

\[ y = x^3 + \frac{C_1 x^2}{2} + C_2, \quad C_1, C_2 \in \mathbb{R}. \]

Because \( C_1 \) is an arbitrary constant, equivalently the general solution is

\[ y = x^3 + C_1 x^2 + C_2. \]

This ODE was inhomogeneous. When we did \( ay'' + by' + cy = 0 \), we got general solutions of the form \( y = c_1 y_1 + c_2 y_2 \). In particular, \( c_1 = 0, c_2 = 0 \) means that \( y = 0 \) is solution of homogeneous linear equation, but will not be a solution for an inhomogeneous linear equation and instead there will be a separate term that is a function of \( x \) or \( t \), here \( x^3 \).

**Example** \( y'' = 1 + (y')^2 \). \( y \) does not occur: only \( y', y'' \) occur. Let \( z = y' \). Then \( z' = 1 + z^2 \). This is \( \frac{1}{1+z^2} z' = 1 \), i.e. \( -1 + \frac{1}{1+z^2} z' = 0 \). This is a separable first order ODE with \( M(x) = -1, N(z) = \frac{1}{1+z^2} \).

\[ H_1(x) = -x, \quad H_2(z) = \arctan z, \quad \text{as } (\arctan z)' = \frac{1}{1+z^2}. \]

Then by method of separable first order ODE,

\[-x + \arctan z = C_1.\]

\( \arctan z = x + C_1 \). Using \( \tan \arctan z = z \), which is true for all \( z \),

\[ z = \tan(x + C_1). \]

\( z = y' \) implies

\[ y'(x) = \tan(x + C_1). \]

Using \( \int \tan x dx = -\ln |\cos x| \) and assuming that \( -\ln |\cos x| = -\ln \cos x \),

\[ y(x) = -\ln \cos(x + C_1) + C_2 = \ln \frac{1}{\cos(x + C_1)} + C_2 = \ln \sec(x + C_1) + C_2. \]

\( -\ln \cos(x + C_1) \) and \( \ln \sec(x + C_1) \) are both acceptable. Usually I try to remove negative signs but sec is less familiar than cos so removing the negative sign has a cost so there is no clear better choice.

**Example of nonlinear equation that has \( y \) but that can be solved by rearranging \( yy'' = (y')^2 \).**

Rearrange:

\[ \frac{yy''}{y'} = \frac{y''}{y'} = \frac{y'}{y}. \]

That is, making assumptions to avoid absolute values,

\[ (\ln y')' = (\ln y)' .\]
Then integrating 
\[ \ln y' = \ln y + C_1. \]
Then, as \( \ln y' - \ln y = \ln \frac{y'}{y} \),
\[ \ln \frac{y'}{y} = C_1. \]
Taking exponentials,
\[ \frac{y'}{y} = e^{C_1}. \]
For \( A = e^{C_1} \),
\[ y' = Ay \quad \Rightarrow \quad y' - Ay = 0. \]
This is a first order linear ODE, \( \mu = e^{\int -Adt} = e^{-At} \). Solution of \( y' - Ay = 0 \)
is, with \( g = 0 \),
\[ y(t) = \frac{1}{\mu(t)} \left( \int \mu(t)g(t) + C_2 \right) = C_2 e^{At}. \]
Therefore, for \( A > 0 \) and for \( C_2 \in \mathbb{R} \), \( y(t) = C_2 e^{At} \) is a solution of \( yy'' = (y')^2 \).
Perhaps if we were careful about absolute values there would not be the restriction on \( A > 0 \): making no assumptions about whether \( A > 0, A = 0, A < 0 \), and for any \( C_2 \in \mathbb{R} \), \( y(t) = C_2 e^{At} \) satisfies \( y' = AC_2 e^{At}, y'' = A^2 C_2 e^{At} \). Then
\[ yy'' = C_2 e^{At} A^2 C_2 e^{At} = A^2 C_2^2 e^{2At}, \]
\[ (y')^2 = (AC_2 e^{At})^2 = A^2 C_2^2 e^{2At}. \]
That is, indeed it is not necessary to assume \( A > 0 \) for \( C_2 e^{At} \) to be a solution. I mention this as one reason why we should remember that when we drop absolute values we are making assumptions, and that to have a solution that applies to every case we may need to keep absolute values.