MAT B44 Integral Curve Examples

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See integralcurvesnb.pdf

The point of these exercises is to match an ODE with an integral curve plot. Usually we work by exclusion: we check that some ODE/plots are not pairs until there is only one possibility left. I am going to explain for each of the plots the information I want you to be able to extract from the plot.

**Plot 1** The easiest information to extract from the plot is (i) where solutions have vertical tangents and (ii) where solutions have horizontal tangents. Vertical tangents correspond to $y'(t) = \infty$ or $y'(t) = -\infty$, and horizontal tangents correspond to $y'(t) = 0$. From the plot, we get that for any $t_0$, a solution through the point $(t_0, 1)$ satisfies $y'(t_0) = \pm\infty$. It is not as obvious but a second piece of information is that the curves are horizontal at $t = 0$. That is, $y'(0) = 0$.

(remember each of the curves is a separate solution, and we are saying that all the solutions have this property.)

From this information we cannot reconstruct $y' = \frac{t^2}{1-y^2}$. For example, $y' = \frac{t}{1-y^2}$ is also consistent with the above information.

**Plot 2** There are several points where integral curves have horizontal tangents, which corresponds to $y'(t) = 0$. For example, the solution through approximately $(-3, 6)$ has a horizontal tangent at this point.

**Plot 3** Horizontal tangent at $y = 0$ and vertical tangent at $t = -1$. That is, if $y$ is the solution through $(t_0, 0)$ then $y'(t_0) = 0$, and also $y'(-1) = \pm\infty$.

**Plot 4** Curves are nearly horizontal when $y$ is large, at least when $t$ is small.

**Plot 5** It is not clear where the curves have horizontal tangents. One vertical tangent we can notice is the curve that passes near to $(-3, -6)$. That is, for this solution we have $y'(-3) = 0$.

**ODE 1**

\[ y' = \frac{t^2}{1-y^2} \]

**ODE 2**

\[ y' = t^2 - y. \]

**ODE 3**

\[ y' = \frac{y^2}{1+t^3}. \]

**ODE 4**

\[ y' = \frac{4t-t^3}{4+y^3}. \]
ODE 5

\[ y' = \frac{4y - 3t}{2t - y} \]

Plot 1: We can exclude ODE 2 because a solution of ODE 2 never has \( y' = \pm \infty \). Plot 1 has vertical tangents when \( y = 1 \), for any \( t \), while ODE 3 has vertical tangents when \( t = -1 \), so we can exclude ODE 3. ODE 4 has vertical tangents when \( 4 + y^3 = 0 \), whereas the only vertical tangents for Plot 1 are when \( y = 1 \), so we can exclude ODE 4. Finally, ODE 5 has a vertical tangent at \((-3, -6)\), whereas Plot 1 does not, so we can exclude ODE 5. Therefore we match ODE 1 and Plot 1.