MAT B44 first order ODE example 5

Jordan Bell
Department of Mathematics, University of Toronto

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\[ y' = \frac{5x-x^2}{5+y^3}. \]

separable equation: \( (5+y^3)y' = 5x-x^2, \)
\[ 5x-x^2 - (5+y^3)y' = 0. \]

\( M(x) = 5x-x^2, \ N(y) = -5-y^3. \)

\( H_1(x) = M(x), \ H_1(x) = \frac{5}{2}x^2 - \frac{x^3}{3}, \)
\[ H_2(y) = -5y - \frac{y^4}{4}. \]

\[ H_1(x) + H_2(y) = C. \]

\[ \frac{5}{2}x^2 - \frac{x^3}{3} - 5y - \frac{y^4}{4} = C. \]

That is, if \( y(x) \) is a solution of the differential equation then there is some \( C \) such that

\[ \frac{5}{2}x^2 - \frac{x^3}{3} - 5y(x) - \frac{y(x)^4}{4} = C. \]

There is no simple way of writing \( y(x) \) explicitly in terms of \( x \).

Euler’s method for IVP \( y(1) = 3. \) Approximate: \( y(1.1), y(1.2). \)

\[ y_0 = 3. \]

\[ y_{n+1} = y_n + hf(t_n, y_n), \quad t_n = t_0 + nh, \quad f(t, y) = \frac{5t-t^2}{5+y^3} \]

\( t_1 = t_0 + h = 1 + 0.1 = 1.1. \)

\[ y_1 = 3 + 0.1 \cdot f(1,3) = 3.0125 \ldots \]

\[ y_2 = 3.0125 \ldots + 0.1 \cdot f(1.1,3.0125 \ldots) = 0.399631 \ldots \]

Using numerically computed solution, \( y(t_1) = y(1.1) = 3.01289 \ldots \) and \( y(t_2) = y(1.2) = 3.02649 \ldots \)

\( y_1 \) is fine as an approximation to \( y(t_1) \) but \( y_2 \) is bad as an approximation to \( y(t_2) \).

Picard iterates:

\[ \phi_0(t) = y_0 = 3. \]

\[ t_0 = 1. \]

\[ \phi_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, \phi_n(s))ds. \]
\[ \phi_1(t) = 3 + \int_1^t f(s, 3)\,ds = 3 + \int_1^t \frac{5s - s^2}{5 + 3s^3}\,ds = 3 + \int_1^t \frac{5s - s^2}{32}\,ds \\
= 3 + \left[ \frac{5s^2}{64} - \frac{s^3}{96} \right]_1^t = 3 + \frac{5t^2}{64} - \frac{t^3}{96} - \left( \frac{5}{64} - \frac{1}{96} \right) = \frac{563}{192} + \frac{5t^2}{64} - \frac{t^3}{96}. \]

\( \phi_2(t) \) has no straightforward way of integrating.