MAT B44 first order ODE example 11

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\[ y'(t) = \frac{t^2}{y(t)(1+t^3)}, \quad y(0) = y_0. \]

\[ f(t, y) = \frac{t^2}{y(1+t^3)}, \quad f_y(t, y) = -\frac{t^2}{y^2(1+t^3)}. \] These are not continuous at \((t_0, y_0)\) if either \(y_0 = 0\) or \(1 + t_0^3 = 0\), i.e. \(t_0^3 = -1\). As \(t_0\) is real, \(t_0^3 = -1\) implies \(t_0 = -1\). There are two other complex number solutions of \(t_0^3 = -1\) but those do not matter because here \(t_0\) is assumed to be a real number.

Therefore, \(f\) and \(f_y\) are not continuous at \((t_0, y_0)\) if either \(t_0 = -1\) or \(y_0 = 0\).

In the Mathematica notebook for this example, the pattern changes from one side to the other of the line \(t_0 = -1\) and from one side to the other of the line \(y_0 = 0\).

This is a separable first order ODE:

\[ \frac{y'}{y} = \frac{t^2}{1+t^3} \iff yy' = \frac{t^2}{1+t^3} \iff -\frac{t^2}{1+t^3} + yy' = 0. \]

\[ M(t) = -\frac{t^2}{1+t^3}, \quad N(y) = y. \]

\[ H_1'(t) = M(t) \quad \Rightarrow \quad H_1(t) = -\frac{1}{3} \ln |1 + t^3|. \]

\[ H_2'(y) = N(y) \quad \Rightarrow \quad H_2(y) = \frac{y^2}{2}. \]

Then a solution \(y\) of the ODE satisfies for some \(C\),

\[-\frac{1}{3} \ln |1 + t^3| + \frac{y(t)^2}{2} = C.\]

Then

\[ y(t)^2 = 2C + \frac{2}{3} \ln |1 + t^3|. \]

Then either \(y = y_1\) or \(y = y_2\) for

\[ y_1(t) = \sqrt{2C + \frac{2}{3} \ln |1 + t^3|}, \quad y_2(t) = -\sqrt{2C + \frac{2}{3} \ln |1 + t^3|}. \]
For IVP \( y(0) = y_0 \), as \( \ln 1 = 0 \),
\[
C = \frac{y_0^2}{2}.
\]

For \( t > -1 \), i.e. for \( t \) to the right of \( t_0 = -1 \), it is the case that \(|1 + t^3| = 1 + t^3\), so
\[
y_1(t) = \sqrt{y_0^2 + \frac{2}{3}\ln(1 + t^3)}, \quad y_2(t) = -\sqrt{y_0^2 + \frac{2}{3}\ln(1 + t^3)}.
\]

Check if both satisfy IVP \( y(0) = y_0 \):
\[
y_1(0) = \sqrt{y_0^2} = |y_0|, \quad y_2(0) = -\sqrt{y_0^2} = -|y_0|.
\]

Either \( y_0 < 0 \), \( y_0 = 0 \), \( y_0 > 0 \). If \( y_0 < 0 \) then \( y_1(0) = |y_0| = -y_0 \) and \( y_2(0) = -|y_0| = y_0 \). Then \( y_1 \) does not satisfy IVP and \( y_2 \) satisfies IVP.

If \( y_0 = 0 \) then \( y_1(0) = |y_0| = 0 \) and \( y_2(0) = -|y_0| = 0 \) so both \( y_1 \) and \( y_2 \) satisfy IVP.

If \( y_0 > 0 \) then \( y_1(0) = |y_0| = y_0 \) and \( y_2(0) = -|y_0| = -y_0 \). Then \( y_1 \) satisfies IVP and \( y_2 \) does not satisfy IVP.

For \( t > -1 \), i.e. for \( t \) to the right of \( t_0 = -1 \), \( t \) belongs to the domain of the above solutions (the domain is the same because they have the same square root) if and only if the argument of the square root is nonnegative. That is, if and only if
\[
y_0^2 + \frac{2}{3}\ln(1 + t^3) \geq 0
\]
i.e.
\[
\frac{2}{3}\ln(1 + t^3) \geq -y_0^2
\]
i.e.
\[
\ln(1 + t^3) \geq -\frac{3y_0^2}{2}
\]
i.e.
\[
1 + t^3 \geq e^{-\frac{3y_0^2}{2}}
\]
i.e.
\[
t^3 \geq -1 + e^{-\frac{3y_0^2}{2}}
\]
i.e., taking the cube root of both sides (which preserves the inequality and makes sense whether \( t \) is positive or negative, this is a general fact),
\[
t \geq \left(-1 + e^{-\frac{3y_0^2}{2}}\right)^{1/3} = A(y_0).
\]

I am defining \( A(y_0) \) this way. This calculation shows that for \( t > -1 \), \( t \) belongs to the domain of the above solutions if and only if \( t \geq A(y_0) \). In other words, since \( t_0 = 0 \) the domain of the above solutions is
\[
[A(y_0), \infty).
\]

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For example for $y_0 = 1$,

$$y_1(t) = \sqrt{1 + \frac{2}{3} \ln(1 + t^3)}, \quad y_2(t) = -\sqrt{1 + \frac{2}{3} \ln(1 + t^3)}.$$

By above work, for IVP $y_0 > 0$, $y_1$ satisfies IVP and $y_2$ does not satisfy IVP. So the solution of the IVP $y(0) = 1$ is

$$y_1(t) = \sqrt{1 + \frac{2}{3} \ln(1 + t^3)}.$$

The domain of $y_1$ is

$$[A(y_0), \infty) = [A(1), \infty).$$

$$A(1) = \left(-1 + e^{-\frac{2}{3}}\right)^{1/3} = (-0.77687\ldots)^{1/3} = -0.919283\ldots$$

Summarizing: the IVP $y' = \frac{t^2}{y(1+t^3)}$, $y(0) = 1$, has the unique solution

$$y_1(t) = \sqrt{1 + \frac{2}{3} \ln(1 + t^3)}$$

with domain

$$[-0.919283\ldots, \infty).$$